ESSAYS IN SOCIAL CHOICE THEORY

A Master's Thesis

by
SELMAN EROL

Department of
Economics
Bilkent University
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SELMAN EROL

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Dr. Semih Koray

Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Tarık Kara

Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Azer Kerimov

Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Dr. Erdal Erel

Director

ABSTRACT

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EROL, Selman

M.A., Department of Economics

Supervisor: Prof. Semih Koray

June 2009

In this thesis, we investigate several issues concerning the class of Maskin monotonic social choice rules. Firstly, given a set of profiles, we find out which Maskin monotonic social choice rules adopt this set as a center. Then we introduce an algorithmic approach to find the self-monotonicities of a Maskin monotonic social choice rule. Moreover, we characterize all binary set operations that preserve Maskin monotonicity. Then we pass to investigating social choice functions, and determine the the domains of impossibility and possibility around a center with respect to a modified Manhattan metric. Finally, we try to reach a necessary and sufficient condition for Nash-implementability of a social choice in terms of neutrality.

Keywords: Social Choice Theory, Maskin Monotonicity, Nash Implementation, Center, Self Monotonicity, Manhattan Metric, Impossibility, Preservation of Maskin Monotonicity, Neutrality.

ÖZET

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EROL, Selman

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Bu tez çalışmamızda, Maskin tekdüze sosyal seçme kurallarının sınıfının çeşitli özelliklerini inceliyoruz. İlk olarak, tercih profillerinden oluşan bir kümenin hangi Maskin tekdüze sosyal seçme kuralları tarafından merkez olarak kabul edildiğini buluyoruz. Daha sonra Maskin tekdüze bir sosyal seçme kuralının öz tekdüzeliklerini bulan bir algoritma sunuyoruz. Ayrıca, Maskin monotonluğu koruyan tüm küme işlemlerinin karakterizasyonunu yapıyoruz. Daha sonra sosyal seçme fonksiyonlarını inceliyoruz ve modifiye edilmiş Manhattan ölçütüne göre bir merkez etrafındaki tanım bölgelerinin imkansızlık veya imkanlılık bölgeleri olup olmadığını belirliyoruz. Son olarak, bir sosyal seçme kuralının Nash-uygulanabilirliği için nötrallik cinsinden gerekli ve yeterli bir koşul bulmaya çalışıyoruz.

Anahtar Kelimeler: Sosyal Seçim, Nash Uygulanabilirlik, Maskin Tekdüzelik, Merkez, Öz Tekdüzelik, Manhattan Ölçütü, İmkansızlık, Maskin Tekdüzeliğin Korunurluğu, Nötrallik.

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CHAPTER 1

INTRODUCTION

In this thesis, we investigate several issues concerning Maskin monotonic social choice rules and social choice functions. Mainly we are looking for the underlying structure of Maskin monotonic social choice rules.

Given that a social choice rule is Maskin monotonic, rather than the total set of preference profiles, a subset of profiles is sufficient to tell the outcomes of the rule throughout the whole domain. The idea results from the notion of a critical profile originating from the work of Koray, Adali, Erol, and Ordulu (2001). Afterwards, Koray and Dogan (2008) introduce the notion of a center, intuitively defined as the smallest set of profiles that is sufficient to characterize the rule. In its formal definition, center is a subset of profiles and it does not provide the information about the outcomes at these profiles. Therefore a natural question arises; which social choice rules have the same center? Having the same center is a classification about social choice rules and may shed further light to the structure of the class of Maskin monotonic social choice rules. The answer is sought in chapter 3.

In chapter 4, we delve deeper into the notion of monotonicity. Maskin monotonicity, as the name tells, is a kind of monotonicity, which is one among many. Koray (2002) introduces monotonicity for social choice rules, and it becomes possible to compare social choice rules with respect to their monotonicities, allowing us to answer what degree of monotonicity a social choice rule has (Koray and Dogan 2008). Smallest monotonicities of a social choice rule is called a self monotonicity, which is proven to be related to Nash-implementability of the rule, by Koray and Dogan (2008). With their approach, it becomes easier to check whether a social choice rule is Nash-implementable, and with this chapter it becomes easier to find the self monotonicities of the rule with an algorithmic approach.

Considering Maskin monotonic social choice rules individually, monotonicities tell us a lot. However, the underlying structure of the class of Maskin monotonic social choice rules is not investigated in detail. The chapters concerning the center and self monotonicity turns out to be telling about this structure. Furthermore, this class could be analyzed from an algebraic point of view. It is clear that under union or intersection, Maskin monotonicity is preserved. Then we may talk of the largest Maskin monotonic subcorrespondence of any social choice rule. Delving deeper into the topic in chapter 5, we try to find all set operations that preserve Maskin monotonicity. Therefore we may talk of various algebraic operations causing partial orders on Maskin monotonic social choice rules allowing for maximal elements and equivalence classes.

In chapter 6, we focus our attention to Mueller-Satterthwaite theorem and the impossibility result. Under full domain and at least three alternatives, it is impossible to find an onto, Maskin monotonic, and non-dictatorial social choice function. Koray and Gurer (2008) give conditions in terms of the domain of the function using Manhattan metric, so that we can get rid of the impossibility result. We modify Manhattan metric to investigate the key element of impossibility by giving different weights to transpositions. Moreover, it turns out that there may occur nested domains of impossibility and possibility. We mainly employ the idea in the proof of Mueller-Satterthwaite theorem provided by Koray, Adali, Erol, and Ordulu (2001).

Lastly in chapter 7, we look for conditions necessary and sufficient for Nash-implementability of Maskin monotonic social choice rules in terms of neutrality. Due to Maskin (1977) and Moore and Repullo (1990), it is well known that a form of monotonic behavior of the social choice rule and the assumption that individuals' veto powers are limited are the key factors in Nash-implementability. It has been proven by Maskin (1977) that Ne-Veto-Power and Maskin monotonicity are sufficient conditions whereas Maskin monotonicity is a necessary condition. Then it is natural to narrow the conditions to arrive at a necessary and sufficient condition, which is indeed achieved by Moore and Repullo (1990), by weaking Ne-Veto-Power and tolerating it with strengthening Maskin monotonicity and assuming a form unanimity. However, we know that neutrality and Maskin monotonicity is also a set of necessary conditions for Nash-implementability. We try to weaken neutrality in order to obtain necessary and sufficient conditions for Nash-implementability in terms neutrality, or at least try to find an appropriate approach in doing so.

CHAPTER 2

PRELIMINARIES

Throughout the thesis, A will denote the finite set of alternatives and N will denote the finite set of individuals. A linear order is a transitive, antisymmetric, and complete binary relation. $\mathcal{L}(A)$ is the set of all linear orders on A. An element of $\mathcal{L}(A)^N$ will be called a preference profile. A social choice function (SCF) is a function $F : \mathcal{L}(A)^N \to A$ and a social choice rule (SCR) is a function $F : \mathcal{L}(A)^N \to 2^A$. For any $a \in A$ and $P \in \mathcal{L}(A)$ the lower contour set of a at P is $L(a, P) := \{b \in A : aPb\}$ and the strict lower contour set of a at P is $L'(a, P) := L(a, P)\backslash\{a\}$.

An SCR F is Maskin monotonic if for any $R, R' \in \mathcal{L}(A)^N$ and for any $a \in A$: $[a \in F(R)$, and for all $i \in N$ $L(a, R_i) \subset L(a, R'_i)$ imply $a \in F(R')]$. Similarly, an SCF F is Maskin monotonic if for any $R, R' \in \mathcal{L}(A)^N$ and for any $a \in A$: $[a = F(R)$, and for all $i \in N$ $L(a, R_i) \subset L(a, R'_i)$ $\forall i \in N$ imply $a = F(R')]$.

For any $R \in \mathcal{L}(A)^N$, the triplet (N, A, R) is a normal form game. A function which associates each normal form game (N, A, R) with a subset of A is called a solution concept.

Consider any abstract set M_i for each $i \in N$, called the strategy space of the agent i. $M = \prod\limits_{i \in N} M_i$ is called the strategy space. Take an onto function $\pi : M \to A$, called the outcome function. Then the pair (M, π) is called a mechanism.

Given a mechanism $G = (M, \pi)$ and a preference profile $R \in \mathcal{L}(A)^N$, $u^R \in \mathcal{L}(M)^N$ is defined as $[\forall i \in N,\ \forall m, m' \in M : m u_i^R m'$ if and only if $\pi(m) R_i \pi(m')$. Then the normal form game associated with the mechanism g is $G[R] = (N, M, u^R)$.

Given an SCR F and a solution concept σ, a mechanism $G = (M, \pi)$ is said to $\sigma-$implement F if for any $R \in \mathcal{L}(A)^N$, one has $\pi(\sigma(G[R])) = F(R)$.

An SCR F is said to be $\sigma-$implementable if there exists a mechanism which $\sigma-$implements F.

CHAPTER 3

WHICH SOCIAL CHOICE RULES HAVE THE SAME CENTER?

For any alternative $a \in A$, the set of preference profiles can be partitioned into equivalence classes with respect to the lower contour sets of a.

Definition. For any alternative $a \in A$, the set of equivalence classes on $\mathcal{L}(A)^N$ with respect the lower contour sets of a is defined as $\rho(a) = \{\{R' \in \mathcal{L}(A)^N : \forall \in N,\ L(a, R_i) = L(a, R'_i)\} : R \in \mathcal{L}(A)^N\}$

Any preference profile belongs to some element of $\rho(a)$ by definition. For some $a \in A$, two elements taken from an equivalence class in the partition $\rho(a)$ induce the same lower contour set for any agent $i \in N$. What follow are the notions of refinement and critical profile which will be very useful in the rest.

Definition. For any $R, R' \in \mathcal{L}(A)^N$ and $a \in A$ we say that R' is an $a-$refinement of R if for all $i \in N$ one has $L(a, R'_i) \subset L(a, R_i)$. If at least one inclusion is strict, then we also say that R' is a strict $a-$refinement of R.

Definition. Let F be a Maskin monotonic SCR. For any $R, R' \in \mathcal{L}(A)^N$ and $a \in A$, R is called an $a-$critical profile of F if $a \in F(R)$ and for any $a-$refinement R' of R other than R, one has $a \notin F(R')$.

The set of all $a-$critical profiles of F is denoted by $C_a(F)$. Since F is Maskin monotonic, it is easy to notice that if R is an $a-$critical profile and R' is such that lower contour sets of a at R and R' are coincident, then R' is also an $a-$critical profile. Then for each alternative $a \in A$ there exist elements of $\rho(a)$, say $S_1, , S_2, .., S_I$ such that $C_a(F) = \bigcup_{i=1}^{I} S_i$.

Definition. Let F be a Maskin monotonic SCR, $a \in A$, and $C_a(F) = \bigcup_{i=1}^{I} S_i$ for some $S_1, S_2, .., S_I \in \rho(a)$. Take a profile R_i from each set S_i. Then the set $\{R_1, R_2, ..., R_I\}$ is called an $a-$center of F. An $a-$center is denoted by $CE_a(F)$, and a set $CE(F) := \bigcup_{a \in A} CE_a(F)$ is called a center of F. (Note that $CE_a(F)$ and $CE(F)$ are not uniquely determined.)

Center can be interpreted as the minimal set of profiles sufficient to identify an SCR. However, since it is not unique for the SCR, it is natural to ask whether a set is a center for different profiles. The answer turns out to be "yes". In this chapter, our aim is to answer the question: "Given a set of profiles $T = \{R^1, R^2, ..., R^q\}$, which Maskin monotonic social choice rules induce T as a center?".

Definition. For any $a \in A$ and $R \in \mathcal{L}(A)^N$, the LCS function of a, $f_a : \mathcal{L}(A)^N \to (2^A)^{|N|}$ is defined as $f_a(R) = (L'(a, R_1), L'(a, R_2), ..., L'(a, R_N))$.

Throughout the rest, let $n = |N|$ for the ease of notation. The function f_a associates each profile $R \in \mathcal{L}(A)^N$, the $n-$tuple of strict lower contour sets of a at R_i. This function is crucial in identifying the link between the given set T and the critical profiles.

Definition. For each $i \in N$, take a subset A_i of A. If there exists an $a-$critical profile R for F with $f_a(R) = (A_1, A_2, ..., A_n)$, then the $n-$tuple $(A_1, A_2, ..., A_n)$ is called an $a-$critical LCS profile for F.

Definition. The set of all $a-$critical LCS profiles for F is called the refined $a-$center of F.

The notation LCS profile is used to refer to the fact that critical LCS profiles are not preference profiles, but a collection of sets, particularly lower contour sets. Refined $a-$center of F is denoted by $RC_a(F)$. Check that although $a-$center was not uniquely determined by F and a, refined $a-$center is.

Definition. Refined center of F is defined as $RC(F) := \prod_{a \in A} RC_a(F)$.

Similarly, refined center is also uniquely determined by F, in contrast with center.

We introduce the following sets, which will be constituting a feasibility argument for refined center. Check that for any $i \in N$, $R \in \mathcal{L}(A)^N$, and $a \in A$, $a \notin f_a(R)_i$. Also recall that an $a-$critical profile cannot be a strict refinement of another one. Hence an $a-$critical LCS profile cannot be a component by component subset of another one. In the following definitions we make use of these ideas.

Definition. $\mathcal{Y}^a = \{(Y_1, Y_2, ..., Y_n) : \forall i \in N,\ Y_i \subset A \backslash \{a\}\}$, $\mathcal{X}^a = \{\mathcal{Y} \subset \mathcal{Y}^a : [X, Y \in \mathcal{Y}, \text{ and } \forall i \in N,\ X_i \subset Y_i, \text{ imply } X = Y]\}$, $\mathcal{X} = \prod_{a \in A} \mathcal{X}^a$.

For the ease of notation, let $\mathcal{F}$ denote the set of all Maskin monotonic SCR's from $\mathcal{L}(A)^N$ to A. Throughout the rest, with a small abuse of notation, we consider RC as a funtion from $\mathcal{F}$ to $\mathcal{X}$.

Proposition 1. *RC is a bijection between $\mathcal{F}$ and $\mathcal{X}$.*

Proof. We will finish the proof in three steps.

i) RC is a well defined function:

Clear.

ii) $\forall F \in \mathcal{F}$, $RC(F) \in \mathcal{X}$:

Assume that there exist $X, Y \in RC_a(F)$ such that for all $i \in N$, $X_i \subset Y_i$, and $X \neq Y$. Let R^X, R^Y be the corresponding $a-$critical profiles. "$X \neq Y$" implies that there exists $j \in N$ with $L'(a, R^X_j) \neq L'(a, R^Y_j)$. Also, [for all

$i \in N$, $X_i \subset Y_i$] implies that [for all $i \in N$, $L'(a, R_j^X) \subset L'(a, R_j^Y)$]. Then R^X is an $a-$refinement of R^Y, but R^Y is an $a-$critical profile. Contradiction.

Thus for any $X, Y \in RC_a(F)$, [for all $i \in N$, $X_i \subset Y_i$] implies $X = Y$. Clearly, $X \in RC_a(F)$ implies $X \subset A \backslash \{a\}$. Thus $RC(F) \in \mathcal{X}$.

iii) For all $X \in \mathcal{X}$, there exists unique $F \in \mathcal{F}$ with $RC(F) = X$:

Let $X \in \mathcal{X}$. Assume that there exist $F, G \in \mathcal{F}$ such that $F \neq G$ and $RC(F) = RC(G) = X$. Then let there exist a profile R such that $F(R) \neq G(R)$. Hence, there exists some alternative in $F(R) \backslash G(R)$, say a. Let R' be an $a-$critical profile for F, which is a $a-$refinement of R. Then,

$$f_a(R') \in RC_a(F) = RC_a(G) \Rightarrow a \in G(R') \Rightarrow a \in G(R),$$

leading to a contradiction. Also check that F_0 defined as $F_0(R) = \{a \in A : \exists$ an $a-$refinement of R' of R with $f_a(R') \in \mathcal{X}^a\}$ is a Maskin monotonic SCR with $RC(F_0) = X$. $\square$

Turning back to our question, given a subset of $\mathcal{L}(A)^N$, say $T = \{R^1, R^2, ..., R^q\}$, which Maskin monotonic SCR's induce T as a center?

Definition. $T_a := \{f_a(R) : R \in T\}$, $U_a^T := 2^{T_a} \cap \mathcal{X}^a$.

Elements of U_a^T are feasible candidates for being $a-$critical LCS profiles in accordance with T being a center.

Definition. Let $f_{a,T} : T \rightarrow T_a$ be the restriction of f_a to T, and $g_{a,T}^X : X \cap T_a \rightarrow T$ be the restriction of $f_{a,T}^{-1}$ to $X \cap T_a$.

Definition. $\mathcal{U}^T = \{\mathcal{X} \in \prod_{a \in A} U_a^T : \forall a \in A$, $g_{a,T}^{X_a}$ has a singleton valued subcorrespondence $h_{a,T}^{X_a}$ such that $\bigcup_{a \in A} Im(h_{a,T}^{X_a}) = T\}$.

Note that $\mathcal{U}^T$ is independent of F.

Proposition 2. *T is a center of F if and only if $RC(F) \in \mathcal{U}^T$.*

Proof. Let T be a center of F. For all $a \in A$ denote $X_a = RC_a(F)$. Now for each $a \in A$, there exists an $a-$center $CE_a(F)$ such that $T = CE(F) = \bigcup_{a \in A} CE_a(F)$. Fix an alternative $a \in A$. Now there exist $S_1, S_2, ..., S_I \in \rho(a)$ such that $C_a(F) = \bigcup_{t=1}^{I} S_t$. Then for each $t \in \{1, 2, ..., I\}$, there exists ${}^tR$ so that $CE_a(F) = \{{}^1R, {}^2R, ..., {}^IR\}$. Clearly $X_a = \{f_a(R) : R \in CE_a(F)\}$. Since $CE_a(F) \subset T$, $X_a \subset T_a$. Also, since ${}^tR$'s are $a-$critical profiles from different equivalence classes (S_i's), we have $X_a \in U_a^T$. Now let $\forall L \in X_a = X_a \cap T_a$; $h_{a,T}^{X_a}(L) = f_a^{-1}(L) \cap CE_a(F)$. Since ${}^tR$'s are from different S_i's, $h_{a,T}^{X_a}$ is singleton valued. It is also clear that $h_{a,T}^{X_a}$ is a subcorrespondence of $g_{a,T}^{X_a}$.

Recall that $\bigcup_{a \in A} CE_a(F) = T$. $\forall R \in T$, $\exists a \in A$ with $R \in CE_a(F)$. $f_a(R) \in X_a \cap T_a$, thus $h_{a,T}^{X_a}(f_a(R)) = f_a^{-1}(f_a(R)) \cap CE_a(F) \supset \{R\}$. Hence, $R \in Im(h_{a,T}^{X_a})$, which implies $T \subset \bigcup_{a \in A} Im(h_{a,T}^{X_a})$. Also since $h_{a,T}^{X_a}(L) \subset CE_a(F) \subset T$, we have $\bigcup_{a \in A} Im(h_{a,T}^{X_a}) \subset T$. Therefore $T = \bigcup_{a \in A} Im(h_{a,T}^{X_a})$. So, $RC(F) = \bigcup_{a \in A} X_a \in \mathcal{U}^T$.

For the converse, consider $F \in \mathcal{F}$ with $RC(F) \in \mathcal{U}^T$. Let $RC(F) = X$, and for all $a \in A$, $RC_a(F) = X_a$. For all $a \in A$, define $\mathcal{V}_a := Im(h_{a,T}^{X_a})$. $X \in \mathcal{U}^T$ implies

$$\bigcup_{a \in A} \mathcal{V}_a = T. \tag{3.1}$$

There exist $S_1, S_2, ..., S_I \in \rho(a)$ such that $C_a(F) = \bigcup_{t=1}^{I} S_t$. We will follow three steps in order to complete the proof.

i) $R \in \mathcal{V}_a$ implies that there exists $t \in \{1, 2, ..., I\}$ with $R \in S_t$:

$R \in \mathcal{V}_a \Rightarrow h_{a,T}^{X_a}(L) = R$ for some $L \in X_a \cap T_a = X_a$. Hence $R \in g_{a,T}^{X_a}(L) = f_{a,T}^{-1}(L) \Rightarrow f_a(R) = L \in X_a = RC_a(F) \Rightarrow R$ is an $a-$critical profile$\Rightarrow R \in CE_a(F) = \bigcup_{i \in I} S_i \Rightarrow R \in S_t$ for some $t \in I$.

ii) $R, R' \in \mathcal{V}_a$, $R \neq R'$ implies that R and R' are from different S_i's:

$R, R' \in \mathcal{V}_a = Im(h_{a,T}^{X_a}) \Rightarrow \exists L, L' \in X_a$ with $L \neq L'$ since $h_{a,T}^{X_a}$ is singleton valued. $h_{a,T}^{X_a}(L) = R$, $h_{a,T}^{X_a}(L') = R' \Rightarrow R \in g_{a,T}^{X_a}(L) = f_{a,T}^{-1}(L)$, $R' \in f_{a,T}^{-1}(L')$

$\Rightarrow L = f_a(R)$, $L' = f_a(R') \Rightarrow f_a(R) \neq f_a(R')$ since $L \neq L'$. Hence, R, R'

are from different S_i's.

iii) $\forall t \in I$, $S_t \cap \mathcal{V}_a \neq \emptyset$.

Take $R \in S_t$. R is an $a-$critical profile, thus $f_a(R) = RC_a(F) = X_a$. $f_a(R) \in X_a$, then let $R' = h^{X_a}_{a,T}(f_a(R))$. This implies that $R \in g^{X_a}_{a,T}(f_a(R)) = f^{-1}_{a,T}(f_{a,T}(R))$, i.e. $f_a(R) = f_a(R')$,hence $R' \in S_t$. Also $R' = h^{X_a}_{a,T}(f_a(R)) \Rightarrow R' \in Im(h^{X_a}_{a,T}) = \mathcal{V}_a$. So, $R' \in S_t \cap \mathcal{V}_a$.

Combining i, ii, iii, we obtain $\mathcal{V}_a$ is an $a-$center for F and hence (3.1) implies $T = \bigcup\limits_{a \in A} \mathcal{V}_a$ is a center for F. $\square$

Corollary. *For any* $T \subset \mathcal{L}(A)^N$, $RC^{-1}(\mathcal{U}^T) = CE^{-1}(T)$.

Proof. Straightforward. $\square$

Corollary. T *is a center for some Maskin monotonic SCR if and only if* $\mathcal{U}^T \neq \emptyset$.

Proof. Straightforward. $\square$

Corollary. *Let* T *be a center for some* F *and* $T' \subset T$*. Then* T' *is a center for some* F'.

Proof. Since T is a center for some F, $\mathcal{U}^T \neq \emptyset$. Consider $X = \prod\limits_{a \in A} X_a \in \mathcal{U}^T$. Let $X'_a = \{M \in X_a : f_a(R) = M \text{ for some } R \in T'\}$. It is clear that $X'_a \in \mathcal{U}^{T'}$. Define $\bar{h}^{X'_a}_{a,T'}(L) := h^{X_a}_{a,T}(L) \cap T'$. Clearly, $\bigcup\limits_{a \in A} Im(\bar{h}^{X'_a}_{a,T'}) = T'$. Let $h^{X'_a}_{a,T'}$ be defined as:

$$h^{X'_a}_{a,T'}(L) = \begin{cases} \bar{h}^{X'_a}_{a,T'}(L) & \text{if } \bar{h}^{X'_a}_{a,T'}(L) \neq \emptyset, \\ \text{an arbitrary } R \in T' \text{ with } f_a(R) = L & \text{if otherwise.} \end{cases}$$

It is straightforward from the construction that $h^{X'_a}_{a,T'}$ is a singleton valued subcorrespondence of $g^{X'_a}_{a,T'}$. Therefore, $\prod\limits_{a \in A} X'_a \in \mathcal{U}^{T'} \Rightarrow \mathcal{U}^{T'} \neq \emptyset$. $\square$

Despite the fact that notation is complicated and difficult to follow, what we are doing is quite simple. In order to explain the process, consider the below table below. Each column is for a profile in T and each row is for an alternative.

A	R^1	R^2	..	..	R^q	
a	$f_a(R^1)$	$f_a(R^2)$	..	..	$f_a(R^q)$	$\rightarrow T_a$ (there are repititions)
b	$f_b(R^1)$	$f_b(R^2)$	..	..	$f_b(R^q)$	$\rightarrow T_b$ (there are repititions)
c	$f_c(R^1)$	$f_c(R^3)$	..	..	$f_c(R^q)$	$\rightarrow T_c$ (there are repititions)
..	..	..	..	..	..	..

Take an alternative a. Note that there are repetitions in the group ($f_a(R^1)$, $f_a(R^2)$, ... , $f_a(R^q)$). Let $T_a = \{f_a(R^i) : R^i \in T\} = \{M_1^a, M_2^a, ..., M_{p_a}^a\}$. Let r_t^a be defined as $r_t^a = \{R^i \in T : f_a(R^i) = M_t^a\}$. In order to explain, w.l.o.g let $f_a(R^1) = f_a(R^2) = M_1^a$, $f_a(R^3) = M_2^a$, ..., $f_a(R^{q-2}) = f_a(R^{q-1}) = f_a(R^q) = M_{p_a}^a$, i.e. $r_1^a = \{R^1, R^2\}$, $r_2^a = \{R^3\}, ..., r_{p_a}^a = \{R^{q-2}, R^{q-1}, R^q\}$. Hence the row of a in the above table is as follows:

$$a \quad \underbrace{R^1, R^2}_{M_1^a} \quad \underbrace{R^3}_{M_2^a} \quad ... \quad \underbrace{R^{q-2}, R^{q-1}, R^q}_{M_{p_a}^a} \qquad \rightarrow T_a \text{ (without repititions)}$$

The tables for other alternatives have different shapes. For T to be a center, we need to choose a Pareto optimal subset M_a of $T_a = \{M_1^a, M_2^a, ..., M_{p_a}^a\}$, in the sense that no pair of M_i^a's we choose can be component by component inclusive. Moreover, we need to choose a single (at least one would be more appropriate but more than one is unnecessary) profile from each r_t^a. Cumulatively after doing this for each alternative, each profile R^i in T should have been chosen at least once. Otherwise, a subset of T would be the center.

The Pareto optimal subsets of T_a's we choose, i.e. M_a corresponds to X_a. Chosen profile R^i's for M_i^a correspond to $a-$critical profiles.

Note 1. $\sum\limits_{a\in A} |T_a| = \sum\limits_{a\in A} |M_a| \geq |T|$.

Corollary. *If T is a center,* $|A|2^{|N|(|A|-1)} \geq |T|$.

Proof. As noted above, $\sum\limits_{a\in A} |T_a| \geq |T|$. Note that $(2^{|A|-1})^{|N|} \geq |T_a|$. Hence,

$$|A|2^{|N|(|A|-1)} \geq |T|.$$

$\square$

Corollary. *$\mathcal{L}(A)^N$ is never a center.*

Proof. If $\mathcal{L}(A)^N$ is a center, we have $|A|2^{|N|(|A|-1)} \geq (|A|!)^{|N|}$, which implies $\sqrt[|N|]{|A|} \geq \frac{|A|!}{2^{|A|-1}}$. Then either $|A| = 2$, $|N| \geq 3$ or $|A| \in \{2, 3\}$, $|N| = 2$. However, $|N| \geq 3, |A| \geq 3$ were assumed in this chapter. $\square$

CHAPTER 4

AN ALGORITHMIC CHARACTERIZATION OF SELF-MONOTONICITY

For this chapter, we relax the domain of the SCR's we consider. We take a set $\mathcal{R} \subset \mathcal{L}(A)^N$, and consider Maskin mootonic SCR's from $\mathcal{R}$ to A. In this chapter, F will generically denote a Maskin monotonic SCR from $\mathcal{R}$ to A.

Definition. A monotonicity of F is a function $h : Gr(F) \to (2^A)^n$ such that for every $(R, a) \in Gr(F)$,

$$[L_i(a, R) \cap h_i(a, R) \subset L_i(a, R'),\ \forall i \in N \text{ implies } a \in F(R')].$$

Corollary. *If h is a monotonicity of F, then any h' with $[h_i(a, R) \subset h'_i(a, R)$, $\forall i \in N,\ a \in A,\ R \in \mathcal{R}]$ is also a monotonicity of F.*

Corollary. *h is a monotonicity of F if and only if $\forall a \in A,\ R, R' \in \mathcal{R}$ with $a \in F(R) \backslash F(R')$, one has $L_i(a, R')^c \cap L_i(a, R) \cap h_i(a, R) \neq \emptyset$.*

Definition. h is a self monotonicity of F if h is a monotonicity of F, and $h' : Gr(F) \to (2^A)^n$ with $h'_i(a, R) \subset h_i(a, R)$, $\forall (i, a, R) \in N \times Gr(F)$, implies $h' = h$.

Corollary. *If h is a self monotonicity of some F, then for all $(i, a, R) \in N \times Gr(F)$, $h_i(a, R) \subset L(a\,, R_i) \backslash \{a\}$.*

Corollary. *Let h and h' be two self monotonicities of F. Take any subset Δ of $Gr(F)$. Define h'' as:*

$$h''_i(a,R) = \begin{cases} h_i(a,R) & \text{if } (a,R) \in \Delta, \\ h'_i(a,R) & \text{if } (a,R) \notin \Delta. \end{cases}$$

Then h'' is also a self monotonicity of F.

In the light of the above corollary, one may claim that, by characterizing a self monotonicity at a single pair $(R,a) \in Gr(F)$, we would have characterized all self monotonicities cumulatively for its domain. With a small abuse of language, from now on, we will say $h(a,R)$ is a self monotonicity and characterize the possible sets for (a,R), which makes h a self monotonicity. Fix some $(\bar{a},\bar{R}) \in Gr(F)$ for the rest of the chapter. Let $\mathcal{R}^* = \{R \in \mathcal{R} : \bar{a} \notin F(R)\}$, $k = |\mathcal{R}^*|$, $L_i = L(\bar{a},\bar{R}_i)\backslash\{a\}$.

Definition. The correspondence $G : N \times A \to \mathcal{R}^*$ is defined as $G(i,b) = \{R \in \mathcal{R}^* : b \in L_i \cap L(\bar{a},R_i)^c\}$.

By Maskin monotonicity of F, $\forall R \in \mathcal{R}^*$, $\exists (i,b) \in N \times A$ with $R \in G(i,b)$.

Definition. For $M \subset \mathcal{R}^*$, define $\mathcal{T}(M) := \{(i,b) \in N \times A : G(i,b) = M\}$.

Let $\mathcal{G} := \{M \subset \mathcal{R}^* : \mathcal{T}(M) \neq \emptyset\}$.

Definition. A pair (f,g) is "nice" if there exists $s \in \{1,2,...,k\}$, $f : \{1,2,...,s\} \to \mathcal{R}^*$ is a 1-1 function, and g is a nonempty valued correspondence $g : \mathcal{R}^*_f \to \{1,2,...,s\}$, where $\mathcal{R}^*_f = \mathcal{R}^*\backslash Im(f)$.

Definition. For any nice pair (f,g), and for any $t \in \{1,2,...,|Dom(f)|\}$, $A_t(f,g) := \{f(t)\} \cup g^{-1}(t)$.

Definition. A class of set of profiles $\bar{A} = \{A_1, A_2, ..., A_s\}$ is "$(f,g)-$feasible" if (f,g) is a nice pair with $s = |Dom(f)|$ and $A_t = A_t(f,g)$, $\forall t \in \{1,2,...,s\}$. $\bar{A}' = \{A'_1, A'_2, ..., A'_{s'}\}$ is "feasible" if there exists a nice pair (f,g) for which $\bar{A}'$ is $(f,g)-$feasible.

Let $\mathcal{F}$ be the set of all feasible set of set of profiles and $\mathcal{F}_{\mathcal{G}} = \mathcal{F} \cap 2^{\mathcal{G}}$.

Definition. A function $\mathcal{H}$ is "good" if $Dom(\mathcal{H}) \in \mathcal{F}_{\mathcal{G}}$ and $Gr(\mathcal{H}) \subset Gr(\mathcal{T})$.

Definition. For every good $\mathcal{H}$, $h^{\mathcal{H}}$ is defined as $h_i^{\mathcal{H}} = \{b \in A : (i,b) \in Range(\mathcal{H})\}, \forall i \in N$.

Theorem 1. *$h(\bar{a}, \bar{R})$ is a self monotonicity if and only if $h(\bar{a}, \bar{R}) = h^{\mathcal{H}}$ for some good $\mathcal{H}$.*

Proof. Let $h(\bar{a}, \bar{R})$ be a self monotonicity. For the ease of notation, let $h_i = h_i(\bar{a}, \bar{R})$. From corollary (1), $h_i \subset L_i, \forall i \in N$. Let $B := \{(i,b) \in N \times A : b \in h_i$ for some $i\}$, and $T := \{M \subset \mathcal{R}^* : G(i,b) = M$ for some $(i,b) \in B\}$.

We claim that each element of T includes a profile which is not included in other elements of T, i.e. $\forall M \in T, \exists R \in \mathcal{R}^*$ such that $R \in M$, and $R \notin M'$ for any $M' \in T$ with $M \neq M'$. Suppose otherwise, then $\exists M_0 \in T$ such that $M_0 \subset \bigcup_{M' \neq M_0, M' \in T} M'$. There exists an element (i', b') of B with $G(i', b') = M_0$. Define h^* as $h^*_{i'} := h_{i'} \backslash \{b'\}$ and $h^*_i = h_i, \forall i \neq i'$. Take any $R \in \mathcal{R}^*$. Since h is a monotonicity, there exists $i \in N, b \in A$ with $b \in h_i \cap L(\bar{a}, R)^c$. If $(i,b) \neq (i', b')$, clearly $b \in h^*_i \cap L(\bar{a}, R)^c$. If $(i,b) = (i', b')$, since $M_0 \subset \bigcup_{M' \neq M_0, M' \in T} M'$, there exists $M'' \in T$ other than M_0 with $R \in M''$. Since $M'' \neq M_0$, there exists $(i'', b'') \neq (i', b')$ with $G(i'', b'') = M''$. Then since $R \in M''$, $b'' \in h_{i''} \cap L(\bar{a}, R)^c$. Note that, $(i'', b'') \neq (i', b') = (i, b)$, hence $b'' \in h_{i''}$ if and only if $b'' \in h^*_{i''}$. Then, $b'' \in h^*_{i''} \cap L(\bar{a}, R)^c$. Therefore, for every $R \in \mathcal{R}^*$, there exists $i \in N$ with $h^*_i \cap L(\bar{a}, R)^c \neq \emptyset$. Hence h^* is also a monotonicity. Note that $h^*_i \subset h_i, \forall i$ and $h^* \neq h$, which is contradiction with the fact that h is a self monotonicity.

Let $s = |T|$. Now that we have proved that each element of T includes a profile which is not included in other elements of T, we have $s \leq |\mathcal{R}^*| = k$. Let $T = \{M_1, M_2, ..., M_s\}$. In the light of the above claim, let $R_1, R_2, ..., R_s$ be profiles so that R_i is only present in M_i. Define $f : \{1, 2, ..., s\} \to \mathcal{R}^*$ as $f(n) = R_n$. Clearly f is a 1-1 function. Let $\mathcal{R}^*_f = \mathcal{R}^* \backslash \{R_1, ..., R_s\}$. Define $g : \mathcal{R}^*_f \to \{1, 2, ..., s\}$ as $g(R) = \{n : R \in M_n\}$. $\forall R \in \mathcal{R}^*_f \subset \mathcal{R}^*$, there exists

$(i, b) \in B$ with $R \in G(i, b) \in T = \{M_1, ..., M_s\}$ since h is a monotonicity. Then g is a nonempty valued correspondence. Thus (f, g) is a nice pair.

Now we will prove that $M_n = \{f(n)\} \cup g^{-1}(n), \forall n \in \{1, 2, ..., s\}$. Take any $R \in \{f(n)\} \cup g^{-1}(n)$. If $R = f(n)$, clearly $R = R_n \in M_n$. If $R \in g^{-1}(n)$, from the definition of g, $n \in g(R)$ implies $R \in M_n$. Then $\{f(n)\} \cup g^{-1}(n) \subset M_n$. For the converse part, take any $R \in M_n$. Recall that $\mathcal{R}^*_f = \mathcal{R}^* \backslash \{R_1, ..., R_s\}$, i.e. $\mathcal{R}^* = \bigcup\limits_{t \in \{1,..,s\}} \{f(t)\} \cup \mathcal{R}^*_f$. If $R \in \bigcup\limits_{t \in \{1,..,s\}} \{f(t)\}$, clearly $R = R_n = f(n)$ since $R \in M_n$. If $R \in \mathcal{R}^*_f$, since $R \in M_n$, $n \in g(R)$, hence $R \in g^{-1}(n)$. Therefore, $M_n \subset \{f(n)\} \cup g^{-1}(n)$, hence $\{f(n)\} \cup g^{-1}(n) = M_n$. In other words, $T = \{M_1, ..., M_s\}$, $s = |Dom(f)|$, $M_t = A_t(f, g)$, where (f, g) is a nice pair. Hence T is (f, g) feasible, thus feasible, i.e. $T \in \mathcal{F}$. Moreover, notice that from the definition of T, there exists $(i, b) \in B$ for every n so that $G(i, b) = M_n$, hence $\mathcal{T}(M_n) \neq \emptyset$ implying that $M_n \in \mathcal{G}$. Therefore, finally we get $T \in \mathcal{F}_{\mathcal{G}}$.

Consider $(i', b'), (i'', b'') \in B$ with $(i', b') \neq (i'', b'')$. If $G(i', b') = G(i'', b'')$, define h^* as $h^*_{i''} = h_{i''} \backslash \{b''\}$ and $h^*_i = h_i$, $\forall i \neq i''$. Clearly h^* is still a monotonicity but $h^*_i \subset h_i$, $\forall i$, contradicting with the fact that h is a self monotonicity. Then $\mathcal{H} : T \to N \times A$ defined as $\mathcal{H}(M) = \{(i, b) \in B : G(i, b) = M\}$ is a well defined function. Clearly $\mathcal{H}(M) \in \mathcal{T}(M)$ and $T \in \mathcal{F}_{\mathcal{G}}$, concluding that $\mathcal{H}$ is good.

What is left to prove for the left implication is that $h_i = \{b \in A : (i, b) \in Range(\mathcal{H})\}$, $\forall i$. But notice that from the definition of $\mathcal{H}$, $Range(\mathcal{H}) = \{(i, b) \in B : G(i, b) = M \text{ for some } M \in T\}$, which is indeed equal to B. $B = \{(i, b) \in N \times A : b \in h_i \text{ for some } i\}$, hence $\{b \in A : (i, b) \in B\} = h_i$.

For the right implication, let $\mathcal{H}$ be good. We will prove that $h^{\mathcal{H}}$ is a self monotonicity. Denote $X = Dom(\mathcal{H})$, where X is $(f, g)-$feasible for some nice pair (f, g), and also denote $s = |Dom(f)| = |Dom(\mathcal{H})|$. (It is trivial that $|Dom(f)| = |Dom(\mathcal{H})|$).

The first step is to prove that $h^{\mathcal{H}}$ is a monotonicity. Take any $R \in \mathcal{R}^*$.

Let $X = \{A_1, A_2, ..., A_s\}$, where $A_t = A_t(f, g) = \{f(t)\} \cup g^{-1}(t)$. It is clear that $\bigcup_{t \in \{1,..,s\}} A_t = \mathcal{R}^*$ the definition of g, hence $R \in A_t$ for some t. Let $(i, b) = \mathcal{H}(A_t) \in \mathcal{T}(A_t)$. We have $G(i, b) = A_t$, hence $R \in G(i, b)$ implying that $b \in L_i \cap L(\bar{a}, R_i)^c$. We also know that $(i, b) = \mathcal{H}(A_t)$, hence $b \in \{b' \in A : (i, b) \in Range(\mathcal{H})\} = h_i^{\mathcal{H}}$. Then, $b \in h_i^{\mathcal{H}} \cap L_i \cap L(\bar{a}, R_i)^c$. Therefore $h_i^{\mathcal{H}} \cap L_i \cap L(\bar{a}, R_i)^c \neq \emptyset$ for every $R \in \mathcal{R}^*$.

The second and final step is to show that $h_i^{\mathcal{H}}$ is a self monotonicity. Assume the contrary: there exists a monotonicity $h^{'} \neq h^{\mathcal{H}}$ with $h_j^{'} \subset h_j^{\mathcal{H}}, \forall j$. There exists a pair (i, b) such that $b \in h_i^{\mathcal{H}} \backslash h_i^{'}$. Then define h^* as $h_i^* = h_i^{\mathcal{H}} \backslash \{b\}$, and $h_j^* = h_j^{\mathcal{H}}, \forall j \neq i$. Clearly, $h_j^{'} \subset h_j^*, \forall j$, hence h^* is also a monotonicity. $b \in h_i$, hence $(i, b) = \mathcal{H}(A_t)$ for some $t \in \{1, 2, .., s\}$. Consider $R = f(t)$. Note that this implies $R \in G(i, b)$. Since $R \in \mathcal{R}^*$ and h^* is a monotonicity, there exists (i', b') such that $b' \in h_{i'}^* \cap L(\bar{a}, R_{i'})^c \subset h_{i'} \cap L(\bar{a}, R_{i'})^c$, implying that $R \in G(i', b')$. But recall that from definitions of f and g, the profiles in the domain of f are present in exactly one of the elements of X. Hence $R \in G(i', b')$ and $R \in G(i, b)$ imply $(i', b') = (i, b)$. But then, $b' \in h_{i'}^* \cap L(\bar{a}, R_{i'})^c$ implying that $b \in h_i^* = h_i^{\mathcal{H}} \backslash \{b\}$ yields a contradiction. $\square$

CHAPTER 5

PRESERVATION OF MASKIN MONOTONICITY

One natural question when investigating Maskin monotonic social choice rules is that whether the union or intersection of two Maskin monotonic social choice rules is still Maskin monotonic. The answer is easily "Yes" in case of union or intersection. We may also ask "Under what binary set operations, like union or intersection, Maskin monotonicity is preserved?". Later on, it will turn out that the class of binary set operations that preserve Maskin monotonicity is nothing more than a very natural class that trivially preserves Maskin monotonicity. In this chapter, we assume that $|N| \geq 3$, $|A| \geq 3$.

Definition. A binary set operation on A is a function $* : 2^A \times 2^A \to 2^A$.

Definition. Given any social choice rules $F, G : \mathcal{L}(A)^N \to 2^A$, $F * G : \mathcal{L}(A)^N \to 2^A$ is defined as [for all $R \in \mathcal{L}(A)^N$, $F * G(R) = F(R) * G(R)$].

Definition. We say that a binary set operation $*$ preserves Maskin monotonicity if, given any two Maskin monotonic social choice rules $F, G : \mathcal{L}(A)^N \to 2^A$, $F * G$ is also Maskin monotonic.

We now prove the following proposition which is in the heart of the characterization of the binary set operations that preserve Maskin monotonicity.

Proposition 3. $*$ *preserves Maskin monotonicity if and only if,* $\forall a \in A$, $\forall W$, X, Y, $Z \subset A$ *one has:*

$$\begin{bmatrix} a \in W \Rightarrow a \in X \\ a \in Y \Rightarrow a \in Z \end{bmatrix} \text{ implies } [a \in W * Y \Rightarrow a \in X * Z]. \tag{5.1}$$

Proof. Take any two Maskin monotonic SCRs $F, G : \mathcal{L}(A)^N \to 2^A$. $\forall R, R' \in \mathcal{L}(A)^N$, $a \in A$ with $L(a, R_i) \subset L(a, R'_i)$, $\forall i$, we have $a \in F(R) \Rightarrow a \in F(R')$ and $a \in G(R) \Rightarrow a \in G(R')$. Then by (5.1), $a \in F * G(R) \Rightarrow a \in F * G(R')$, hence $F * G$ is Maskin monotonic. Conversely suppose that there exists an operation $*$ such that it preserves Maskin monotonicity but does not satisfy (5.1). Then there exists $a \in A$, $W, X, Y, Z \subset A$, such that $(W \cap \{a\}) \subset (X \cap \{a\})$, $(Y \cap \{a\}) \subset (Z \cap \{a\})$, $a \in W * Y$, $a \notin X * Z$. In order to obtain a contradiction, we will construct specific Maskin monotonic social choice rules $F, G : \mathcal{L}(A)^N \to 2^A$, such that there exists $R, R' \in \mathcal{L}(A)^N$ with $L(a, R_i) \subset L(a, R'_i)$, $F(R) = W$, $F(R') = X$, $G(R) = Y$, $G(R') = Z$.

For the ease of notation, let $A = \{a, x_1, x_2, ..., x_s\}$, $s \geq 2$. Fix some $P \in \mathcal{L}(A)^N$, define R, R' as:

$$R_1 : x_1 R_1 x_2 R_1 ... R_1 x_s R_1 a,$$

$$R_2 : x_s R_2 x_1 R_2 x_2 R_2 ... R_2 x_{s-1} R_2 a,$$

$$R_3 : x_{s-1} R_3 x_s R_3 x_1 R_3 x_2 R_3 ... R_3 a,$$

$$R_i = P, \ \forall i \notin \{1, 2, 3\},$$

$$R'_1 : a R'_1 x_1 R'_1 x_2 R'_1 ... R'_1 x_s,$$

$$R'_2 : x_1 R'_2 x_2 R'_2 ... R'_2 x_s R'_2 a,$$

$$R'_3 : x_s R'_3 x_1 R'_3 x_2 R'_3 ... R'_3 a,$$

$$R'_i = P, \ \forall i \notin \{1, 2, 3\}.$$

Notice that for any $x \in A$ with $x \neq a$, neither R, nor R' is an $x-$refinement of the other. Hence Maskin monotonicity of F and G does not imply any inclusion between $F(R)$ and $F(R')$, and between $G(R)$ and $G(R')$, except $[a \in F(R) \Rightarrow a \in F(R')]$ and $[a \in G(R) \Rightarrow a \in G(R')]$. Then it is natural to define F and G as follows: For every $x \neq a$:

$$\text{if } x \in W, x \in X : RC_x(F) = \{f_x(R), f_x(R')\},$$

$$\text{if } x \in W, x \notin X : RC_x(F) = \{f_x(R)\},$$

$$\text{if } x \notin W, x \in X : RC_x(F) = \{f_x(R')\},$$

$$\text{if } x \notin W, x \notin X : RC_x(F) = \varnothing.$$

For a :

$$\text{if } a \in W : RC_a(F) = \{f_a(R)\},$$

$$\text{if } a \notin W, a \in X : RC_a(F) = \{f_a(R')\},$$

$$\text{if } a \notin X, C_a(F) = \varnothing.$$

For every $x \neq a$:

$$\text{if } x \in Y, x \in Z : RC_x(G) = \{f_x(R), f_x(R')\},$$

$$\text{if } x \in Y, x \notin Z : RC_x(G) = \{f_x(R)\},$$

$$\text{if } x \notin Y, x \in Z : RC_x(G) = \{f_x(R')\},$$

$$\text{if } x \notin Y, x \notin Z : RC_x(G) = \varnothing.$$

For a :

$$\text{if } a \in Y : RC_a(G) = \{f_a(R)\},$$

$$\text{if } a \notin Y, a \in Z : RC_a(G) = \{f_a(R')\},$$

$$\text{if } a \notin Y, RC_a(G) = \varnothing.$$

Now F and G are well defined monotonic SCRs with centers defined as above. Note that $F(R) = W$, $F(R') = X$, $G(R) = Y$, $G(R') = Z$. Then $a \in W * Y = F * G(R)$, $a \notin X * Z = F * G(R')$, although $L(a, R_i) \subset L(a, R'_i)$, $\forall i$. Thus $F * G$ is not Maskin monotonic. Contradiction with the assumption that $*$ preserves Maskin monotonicity. $\square$

In the preceding parts, we will characterize binary operations $* : 2^A \times 2^A \to 2^A$ which satisfy (5.1).

Definition. For any $* : 2^C \times 2^C \to 2^C$, $U, V \subset C$, $a \in C$,

$$\Upsilon(a, U, V, *) = \begin{bmatrix} & \{a\} \cap (V * V) & \\ \{a\} \cap (U * V) & & \{a\} \cap (V * U) \\ & \{a\} \cap (U * U) & \end{bmatrix}.$$

Proposition 4. *If $*$ satisfies (5.1), for any $(U, V), (U', V')$ with $\{a\} \cap U = \{a\} \cap U'$ and $\{a\} \cap V = \{a\} \cap V'$, we have $\Upsilon(a, U, V, *) = \Upsilon(a, U', V', *)$.*

Proof. $a \in U \Leftrightarrow a \in U'$, $a \in V \Leftrightarrow a \in V'$. Then by (5.1), $a \in U * V \Leftrightarrow a \in U' * V'$, $a \in V * U \Leftrightarrow a \in V' * U'$, $a \in U * U \Leftrightarrow a \in U' * U'$, and $a \in V * V \Leftrightarrow a \in V' * V'$. This means that $\Upsilon(a, U, V, *) = \Upsilon(a, U', V', *)$. $\square$

Define $*_a : 2^{\{a\}} \times 2^{\{a\}} \to 2^{\{a\}}$ where $X *_a Y = X * Y$. Then by the above proposition, we have $\Upsilon(a, U, V, *) = \Upsilon(a, U \cap \{a\}, V \cap \{a\}, *)$ which is equal to $\Upsilon(a, U \cap \{a\}, V \cap \{a\}, *_a)$. Thus for any $U, V \subset A$, $(U * V) \cap \{a\} = (U \cap \{a\}) *_a (V \cap \{a\})$. Therefore,

$$U * V = \bigcup_{a \in A} (U * V) \cap \{a\} = \bigcup_{a \in A} (U \cap \{a\}) *_a (V \cap \{a\}).$$

Definition. $*_a^{MM} = \{*_a^1, *_a^2, *_a^3, *_a^4, *_a^5, *_a^6\}$, where

$$*_a^1 = \begin{bmatrix} & \emptyset & \\ \emptyset & & \emptyset \\ & \emptyset & \end{bmatrix}, *_a^2 = \begin{bmatrix} & \emptyset & \\ \emptyset & & \emptyset \\ & \{a\} & \end{bmatrix}, *_a^3 = \begin{bmatrix} & \emptyset & \\ \{a\} & & \emptyset \\ & \{a\} & \end{bmatrix},$$

$$*_a^4 = \begin{bmatrix} & \emptyset & \\ \emptyset & & \{a\} \\ & \{a\} & \end{bmatrix}, *_a^5 = \begin{bmatrix} & \emptyset & \\ \{a\} & & \{a\} \\ & \{a\} & \end{bmatrix}, *_a^6 = \begin{bmatrix} & \{a\} & \\ \{a\} & & \{a\} \\ & \{a\} & \end{bmatrix}.$$

Proposition 5. *If $*$ satisfies (5.1) and $a \in A$, $\Upsilon(a, \{a\}, \emptyset, *_a) \in *_a^{MM}$.*

Proof. By (5.1),

$$\begin{bmatrix} a \in \{a\} \Rightarrow a \in \{a\} \\ a \in \emptyset \Rightarrow a \in \emptyset \\ a \in \emptyset \Rightarrow a \in \{a\} \end{bmatrix} \Rightarrow \begin{bmatrix} a \in \emptyset * \emptyset \Rightarrow a \in \{a\} * \{a\} \\ a \in \emptyset * \emptyset \Rightarrow a \in \emptyset * \{a\} \\ a \in \emptyset * \emptyset \Rightarrow a \in \{a\} * \emptyset \\ a \in \{a\} * \emptyset \Rightarrow a \in \{a\} * \{a\} \\ a \in \emptyset * \{a\} \Rightarrow a \in \{a\} * \{a\} \end{bmatrix}.$$

It is then clear that $\Upsilon(a, \{a\}, \emptyset, *_a) \in *_a^{MM}$. $\square$

Corollary. *$*$ preserves Maskin monotonicity if and only if $\forall U, V \subset A$,*

$$U * V = \bigcup_{a \in A} (U \cap \{a\}) *_a (V \cap \{a\}),$$

where $\forall a \in A$, $\Upsilon(a, \{a\}, \emptyset, *_a) \in *_a^{MM}$.

Proof. We have already proved that $*$ preserves Maskin monotonicity if and only if $*$ satisfies (5.1), and if $*$ satisfies (5.1), $*$ satisfies the condition in the corollary. The converse is rather trivial. $\square$

We should note that $*_a^1$ corresponds to not choosing a in any case, $*_a^2$ corresponds to intersection, $*_a^3$ corresponds to left operator, $*_a^4$ corresponds to right operator, $*_a^5$ corresponds to union and $*_a^6$ corresponds to constantly choosing a. It is interesting that these were the natural candidates at first glance that preserve Maskin monotonicity. It turned out that considering $*$ alternative by alternative, union of these operations turned out to be the exact characterization of the operations that preserve Maskin monotonicity.

CHAPTER 6

NESTED DOMAINS OF IMPOSSIBILITY AND POSSIBILITY

Under the assumption that there exists at least three alternatives, the well-known Mueller-Satterthwaite Theorem states that an SCF $F : \mathcal{L}(A)^N \to A$ is onto and Maskin monotonic if and only if it is dictatorial. To put it in other words, it is impossible to find an SCF which is Maskin monotonic, onto, and non-dictatoral if there are at least three alternatives. By relaxing the full domain assumption and allowing the society to choose from only a subset of preference profiles, we can get rid of this impossibility result. A domain of impossibility is a subset D of $\mathcal{L}(A)$ where an SCF $F : D^N \to A$ is onto and Maskin monotonic if and only if it is dictatorial, under at least three alternatives. Conversely, a domain of possibility is a subset D' of $\mathcal{L}(A)$ where there exists a non-dictatorial SCF $F : D'^N \to A$ which is onto and Maskin monotonic.

Since $\mathcal{L}(A)$ is the largest domain possible and is a domain of impossibility, it initially gives rise to the idea that, roughly speaking, domains of impossibility are larger than domains of possibility. Indeed, if we consider only domains consisting of a center profile and all profiles within a certain radius of this center with respect Manhattan metric, Koray and Gurer (2008) prove that a domain is a domain of impossibility if and only if it has radius more than $|A|$. This is an elegant result distinguishing two types of domains with a precise border.

However, the Manhattan metric directly counts the minimal number of transpositions to obtain a profile from the other, particularly gives equal weight to each transposition. From this perspective, we miss out whether some transpositions are essential in impossibility. Moreover, it turns out that this property of Manhattan metric is the main reason behind the result that distinguishes two types of domains with a strict condition, namely having radius less than $|A|$ or otherwise.

We define a modified Manhattan metric, adding different numbers for each transposition, and try to identify the essential reason of impossibility. Moreover, it turns out that domains of possibility and impossibility can be nested consecutively.

Denote $|A| = n$. Let $\mathbf{p} = (p_1, p_2, ..., p_{n-1})$ be a vector of positive real numbers. Different than Manhattan metric, we will add p_i instead of 1 when we transpose the alternatives in the i^{th} and $(i+1)^{th}$ places of a linear order (thinking it as a column with most preferred at the top).

Example 1. For $\mathbf{p}$ =$(2, \sqrt{3})$;

$i)$ $d^{\mathbf{p}}(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a \\ c \\ b \end{bmatrix}) = \sqrt{3}$

$ii)$ $d^{\mathbf{p}}(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} c \\ b \\ a \end{bmatrix}) = \min\{2 + \sqrt{3} + 2, \sqrt{3} + 2 + \sqrt{3}\} = 2\sqrt{3} + 2.$

Formally:

Definition. For given $\mathbf{p} = (p_1, p_2, ..., p_{n-1}) \in \mathbb{R}^{n-1}_{++}$, and $P_1, P_2 \in \mathcal{L}(A)$, $d^{\mathbf{p}}(P_1, P_2) := \min\{\sum_{i=1}^{f} p_{T_i} : T = (T_1, ..., T_f) \in \{1, 2, ..., n-1\}^f$ so that after consecutive transpositions of T_i^{th} and $(T_i + 1)^{th}$ elements of P_1, we reach $P_2\}$.

Firstly, we shall prove that this is a well-defined metric:

Proposition 6. *For given* $\mathbf{p} = (p_1, p_2, ..., p_{n-1}) \in \mathbb{R}^{n-1}_{++}$*,* $d^{\mathbf{p}}$ *is a metric on* $\mathcal{L}(A)$*.*

Proof. *i)* It is clear that $\infty > d^{\mathbf{p}}(P_1, P_2) \geq 0$, $\forall P_1, P_2 \in \mathcal{L}(A)$.

ii) Since $p_i > 0$, $d^{\mathbf{p}}(P_1, P_2) = 0$ iff $P_1 = P_2$, $\forall P_1, P_2 \in \mathcal{L}(A)$.

iii) By just considering the transpositions from P_1 to P_2 in the inverse order, it is easy to note that $d^{\mathbf{p}}(P_1, P_2) \geq d^{\mathbf{p}}(P_2, P_1)$. But then similarly, $d^{\mathbf{p}}(P_2, P_1) \geq d^{\mathbf{p}}(P_1, P_2)$, hence $d^{\mathbf{p}}(P_1, P_2) = d^{\mathbf{p}}(P_2, P_1)$, $\forall P_1, P_2 \in \mathcal{L}(A)$.

iv) Consider the transpositions from P_1 to P_2, and from P_2 to P_3. If we add the transpositions, we reach from P_1 to P_3, but it need not be the shortest path. Hence $d^{\mathbf{p}}(P_1, P_2) + d^{\mathbf{p}}(P_2, P_3) \geq d^{\mathbf{p}}(P_1, P_3)$. □

Note that for a Maskin monotonic SCF, being onto is equivalent to being unanimous under full domain. Hence we will recast Mueller-Satterthwaite theorem by replacing the condition of ontoness with unanimity in order to be consistent with limited domains.

Definition. Let $n \geq 3$.

i) $D \subset \mathcal{L}(A)$ is called a domain of impossibility if any Maskin monotonic and unanimous SCF $F : D^N \to A$ is dictatorial.

ii) $D' \subset \mathcal{L}(A)$ is called a domain of possibility if there exists a non-dictatorial, Maskin monotonic, and unanimous SCF $F : D'^N \to A$.

Theorem 2. *Let* $n \geq 3$ *and* $p_n = \infty$*. For given* $\mathbf{p} = (p_1, p_2, ..., p_{n-1}) \in \mathbb{R}^{n-1}_{++}$*,* $\bar{P} \in \mathcal{L}(A)$*, and* $t \in \{1, ..., n-1\}$*,*

i) $\sum_{i=1}^{i=t} p_i \leq r < p_2 + \sum_{i=1}^{i=t} p_i \Rightarrow B_r^{d^{\mathbf{p}}}(\bar{P})$ *is a domain of possibility.*

ii) $p_2 + \sum_{i=1}^{i=t} p_i \leq r < \sum_{i=1}^{i=t+1} p_i \Rightarrow B_r^{d^{\mathbf{p}}}(\bar{P})$ *is a domain of impossibility..*

Proof. Let $\bar{P} = \begin{bmatrix} x_1 \\ .. \\ x_n \end{bmatrix}$.

$i)$ Construct $F : B_k^{d\mathrm{P}}(\bar{P}) \to A$ as

$$F(R) = \{ \begin{array}{ll} x_i & \text{if } i \neq t+1 \text{ and } x_i \text{ is at the top in } R_1 \\ x_{t+1} & \text{if } x_{t+1} \text{ is at the top in } R_1 \text{ and } x_{t+1} R_2 x_1 \\ x_1 & \text{if } x_{t+1} \text{ is at the top in } R_1 \text{ and } x_1 R_2 x_{t+1} \end{array}$$

It is clear that F is unanimous and non-dictatorial since $\bar{R} = \begin{bmatrix} x_{t+1} & x_1 & .. \\ x_1 & x_2 & \\ .. & .. & \end{bmatrix}$ $\in B_r^{d\mathrm{P}}(\bar{P})^N$, and $F(\bar{R}) = x_1 \neq x_{t+1}$. What is left is to prove that F is Maskin monotonic. For some $R \in B_r^{d\mathrm{P}}(\bar{P})^N$, let $F(R) = x_i$. If $i \neq 1, t+1$, then by the construction, one must have x_i is at the top in R_1. Therefore, $F(R') = x_i$ for any $R' \in B_r^{d\mathrm{P}}(\bar{P})^N$ with $L(x_i, R_j) \subset L(x_i, R'_j)\ \forall j$.

If $F(R) = x_1$ and x_1 is at the top in R_1 same method applies. If $F(R) = x_1$ and x_1 is not at the top in R_1, then it must be the case that x_{t+1} is at the top in R_1 and $x_1 R_2 x_{t+1}$. Note that $R_1 \in B_r^{d\mathrm{P}}(\bar{P})$ and x_{t+1} is at the top in R_1, but it takes at least $\sum_{i=1}^{i=t} p_i$ to take x_{t+1} to the top and $k < p_2 + \sum_{i=1}^{i=t} p_i$. Therefore, x_1 must be in the second place in R_1. Take any $R' \in B_r^{d\mathrm{P}}(\bar{P})^N$ with $L(x_1, R_j) \subset L(x_1, R'_j)\ \forall j$. Then either x_1 is at the top in R'_1 or x_{t+1} is at top with x_1 in the second place. Also x_1 must be at the top in R'_2. Therefore $F(R') = x_1$.

If $F(R) = x_{t+1}$, it is clear from the construction that $F(R') = x_{t+1}$ for any $R' \in B_r^{d\mathrm{P}}(\bar{P})$ with $L(x_{t+1}, R_j) \subset L(x_{t+1}, R'_j)\ \forall j$. Therefore F is Maskin monotonic, implying that $B_r^{d\mathrm{P}}(\bar{P})$ is a domain of possibility.

$ii)$ Assume that $p_2 + \sum_{i=1}^{i=t} p_i \leq r < \sum_{i=1}^{i=t+1} p_i$ and $F : B_r^{d\mathrm{P}}(\bar{P})^N \to A$ is a unanimous, Maskin monotonic SCF. We will prove that it is dictatorial.

Note that if $P \in B_r^{d\mathrm{P}}(\bar{P})$, then $x_{t+2}, x_{t+3},..$ cannot be at the top in P. Take any $k \in \{1, .., t+1\}$. (If $k = 1$, replace x_1 with x_2 and x_2 with x_3 in the remaining part. If $k = 2$,replace x_2 with x_3 in the remaining part.)Take x_k as down as as possible keeping all other alternatives in the same order in $\bar{P}$, so that the new ordering is still in $B_r^{d^{\mathrm{P}}}(\bar{P})$, and let this new ordering be S^k. Also let

$$P^k = \begin{bmatrix} x_k \\ x_1 \\ x_2 \\ .. \end{bmatrix} \in B_r^{d^{\mathrm{P}}}(\bar{P}), \text{ and consider } R^k = \begin{bmatrix} P^k & P^k & .. & P^k \end{bmatrix} \in B_r^{d^{\mathrm{P}}}(\bar{P})^N.$$

By unanimity, $F(R^k) = x_k$. Now, starting with R^k, column by column, take x_k as down as possible so that x_k is still chosen and the new profile is still in the domain, keeping all other orderings the same. Let the final profile be R'^k. At least one column of R'^k should be P^k, otherwise x_1 would be chosen by unanimity. Without loss of generality, let $R_j'^k = P^k$ for some j. Now, take any $i \in N$ such that $R_i'^k \neq P^k$ and $R_i'^k \neq S^k$, if such i exists. Let x_m be just below x_k in $R_i'^k$. Since $R_i'^k \neq S^k$, by switching x_m and x_k in $R_i'^k$, the new ordering

$$R_i''^k \text{ is still in } B_r^{d^{\mathrm{P}}}(\bar{P}). \text{ It is also clear that } P'^k = \begin{bmatrix} x_1 \\ x_k \\ x_2 \\ .. \end{bmatrix} \in B_r^{d^{\mathrm{P}}}(\bar{P}). \text{ Then}$$

consider $R''^k = [\ ..\ \ R_{j-1}'^k\ \ P'^k\ \ R_{j+1}'^k\ \ ..\ \ R_{i-1}'^k\ \ R_i''^k\ \ R_{i+1}'^k\ \ ..\ ] \in B_r^{d^{\mathrm{P}}}(\bar{P})^N$. By the definition of R'^k, $F(R''^k) \neq x_k$, then by Maskin monotonicity, $F(R''^k)$ should be both x_1 and x_m, leading to a contradiction. Therefore, for any $i \in N$, $R_i'^k = P^k$ or $R_i'^k = P^k$. Without loss of generality, let $R_1'^k, ..., R_l'^k = P^k$ and $R_{l+1}'^k, ..., R_N'^k = P^k$. We have already proved that $l \geq 1$. Now assume that

$$l \geq 2. \text{ Let } P''^k = \begin{bmatrix} x_2 \\ x_k \\ x_1 \\ .. \end{bmatrix} \text{ and } R'^{k,j} \text{ be defined as}$$

$$[\ R_1'^k\ \ R_2'^k\ \ ..\ \ R_{j-1}'^k\ \ P''^k\ \ P''^k\ \ ..\ \ P''^k\ \ R_{l+1}'^k\ \ R_{l+2}'^k\ \ ..\ \ R_N'^k\ ].$$

We will prove by induction that $F(R'^{k,j}) = x_k$ for every $j \geq 2$. The initial step is $F(R'^{k,l}) = x_k$.Maskin monotonicity easily implies that $F(R'^{k,l}) = x_2$

or x_k. Note that $\begin{bmatrix} x_k \\ x_2 \\ x_1 \\ x_3 \\ .. \end{bmatrix} \in B_r^{d^P}(\bar{P})$. Now consider first three rows of columns $l-1$ and l. Keep everything else fixed.

$$\begin{bmatrix} x_k & x_k \\ x_1 & x_1 \\ x_2 & x_2 \end{bmatrix} \xrightarrow{F} x_k \Rightarrow \begin{bmatrix} x_k & x_k \\ x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_k \Rightarrow \begin{bmatrix} x_1 & x_k \\ x_k & x_2 \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_k \text{ or } x_1.$$

If $\begin{bmatrix} x_1 & x_k \\ x_k & x_2 \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_k$, then $\begin{bmatrix} x_1 & x_k \\ x_k & x_1 \\ x_2 & x_2 \end{bmatrix} \xrightarrow{F} x_k$, which is a contradiction with the definition of R'^k.

Hence $\begin{bmatrix} x_1 & x_k \\ x_k & x_2 \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_1 \Rightarrow \begin{bmatrix} x_1 & x_2 \\ x_k & x_k \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_1 \Rightarrow \begin{bmatrix} x_k & x_2 \\ x_1 & x_k \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_1$ or x_k.

But also we know that $\begin{bmatrix} x_k & x_k \\ x_1 & x_1 \\ x_2 & x_2 \end{bmatrix} \xrightarrow{F} x_k \Rightarrow \begin{bmatrix} x_k & x_2 \\ x_1 & x_k \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_2$ or x_k.

Therefore $\begin{bmatrix} x_k & x_2 \\ x_1 & x_k \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_k$, i.e. $F(R'^{k,l}) = x_k$. Inductively by similar arguments, $F(R'^{k,j}) = x_k$ for every $j \geq 2$.

Now we know that $F(R'^{k,2}) = x_k$. However, x_2 is unanimous in $R'^{k,1}$, hence $F(R'^{k,1}) = x_2$. Now consider the first two columns and three rows of $R'^{k,2}$. $\begin{bmatrix} x_k & x_2 \\ x_1 & x_k \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_k$ and $\begin{bmatrix} x_2 & x_2 \\ x_k & x_k \\ x_1 & x_1 \end{bmatrix} \xrightarrow{F} x_2$.

$$\begin{bmatrix} x_k & x_2 \\ x_1 & x_k \\ x_2 & x_1 \end{bmatrix} \xrightarrow{F} x_k \Rightarrow \begin{bmatrix} x_k & x_2 \\ x_2 & x_k \\ x_1 & x_1 \end{bmatrix} \xrightarrow{F} x_k \Rightarrow \begin{bmatrix} x_k & x_2 \\ x_2 & x_1 \\ x_1 & x_k \end{bmatrix} \xrightarrow{F} x_k \text{ or } x_1.$$

Similarly as before, x_k contradicts with the definition of R'^k.

Then, $\begin{bmatrix} x_k & x_2 \\ x_2 & x_1 \\ x_1 & x_k \end{bmatrix} \xrightarrow{F} x_1 \Rightarrow \begin{bmatrix} x_2 & x_2 \\ x_k & x_1 \\ x_1 & x_k \end{bmatrix} \xrightarrow{F} x_1$.

However, $\begin{bmatrix} x_2 & x_2 \\ x_k & x_k \\ x_1 & x_1 \end{bmatrix} \xrightarrow{F} x_2 \Rightarrow \begin{bmatrix} x_2 & x_2 \\ x_k & x_1 \\ x_1 & x_k \end{bmatrix} \xrightarrow{F} x_2$. Contradiction. Therefore $l = 1$.

This means that for every $k \in \{1, 2, ..., t+1\}$, $\exists i_k \in N$ with $F(\bar{R}^k) = x_k$ where $\bar{R}^k_{i_k} = P^k$ and $\bar{R}^k_j = S^k$ $\forall j \neq i_k$. It is clear that $i_{k_1} = i_{k_2}$ $\forall k_1, k_2 \in \{1, 2, ..., t+1\}$. Then $i_{k_1} \equiv i$ is a dictator. $\square$

Particularly, if p_2 is the largest of all p_i's then we obtain the largest possible domain in terms of a center and a radius around it. Here, largest means that it contains all other domains of possibility that we consider. On the other hand, if p_2 is the smallest of all p_i's, then we obtain many nested domains of impossibility and possibility. This is a rather strange result meaning that an increment in the freedom of people could lead to feasibility of socially desirable outcomes, as well as it could lead to infeasibility of socially desirable outcomes depending on the current state.

CHAPTER 7

NASH-IMPLEMENTATION AND NEUTRALITY

7.1 NEUTRALITY VS NO-VETO-POWER

By Maskin's well known theorem, we know that no-veto-power (NVP) plus Maskin monotonicity is a sufficient condition for Nash-implementability, as well as Neutrality plus Maskin monotonicity is. Moreover, Maskin monotonicity is a necessary condition. Then a natural task is to narrow the conditions to derive a necessary and sufficient condition. From the NVP version of the theorem, the task is achieved by Moore and Repullo (1990). They, in some sense weaken NVP to derive a necessary and sufficient condition, but they tolerate what they lose with this relaxation, by strengthening Maskin monotonicity and including some kind of unanimity. This chapter is a first attempt to derive a necessary and sufficient condition in terms of weakening Neutrality.

But why do we have the idea that Neutrality can be weakened in a meaningful way to arrive at a necessary and sufficient condition ? The well-known Maskin-Wind mechanism is also used to prove that Neutrality plus Maskin monotonicity is a sufficient condition. But the only role of Neutrality in the proof is to allow for a transposition of two alternatives. Then indeed, instead of neutral SCRs, which is a very restrictive class, the SCRs which are "neutral" under transpositions are also Nash-implementable if they are also Maskin monotonic.

Mainly, we will be trying to find a subset of all permutations on the alternatives, which allows us to get a necessary and sufficient condition. We will be keeping Maskin monotonicity as a fixed condition for all the cases we consider, although we must admit that this makes our approach weaker, since being a necessary condition does not imply being a part of any group of necessary and sufficient conditions. Moreover, Moore and Repullo (1977)also do not take Maskin monotonicity as a condition in their statement, but modify it. However, as said before, this is a first attempt, trying to make the analysis and its difficulties clearer, hopefully leading the way to a genuine characterization.

7.2 WEAKINING NEUTRALITY

We will working under the full domain assumption: $\mathcal{R} = \mathcal{L}(A)^N$. Let $\mathcal{M}$ denote the class of all Maskin monotonic SCRs $F : \mathcal{R} \to A$, and $\mathcal{I} \subset \mathcal{M}$ denote the class of all Nash-implementable SCRs $F : \mathcal{R} \to A$.

Definition. A permutation on A is a bijection $\sigma : A \to A$. Denote the set of all permutations on A by $\mathcal{P}$.

We will consider only the permutations on A, yet call them only permutations. For every $R \in \mathcal{R}$, and $\sigma \in \mathcal{P}$, with a small abuse of notation, denote $\sigma(R) \in \mathcal{R}$ as the profile defined as [$\forall a, b \in A$, $\forall i \in N$, $a\sigma(R)_i b$ if and only if $\sigma^{-1}(a) R_i \sigma^{-1}(b)$], or equivalently [$\forall a, b \in A$, $\forall i \in N$, aR_ib if and only if $\sigma(a)\sigma(R)_i\sigma(b)$]. Denote σ_0, the trivial permutation $\sigma_0(a) = a$, $\forall a \in A$.

Roughly speaking, a neutrality of an SCR is a permutation that satisfies the condition in the neutrality definition of an SCR. Throughout the chapter, we will be working in the class $\mathcal{M}$. We will try different approaches for the characterization of Nash-implementation from the viewpoint of neutrality.

There are two main categories. First is SCR-independent, and second is SCR-dependent. More precisely, we will try to find a class of permutations independent of the function in question, or dependent upon the function, which constitutes an if and only if condition by being employed as neutralities of the function. We will make a rigorous analysis in the SCR-independent case: we will consider three main alternatives: the desired class of alternatives will be defined globally meaning that it is independent of either the profile or the alternative in question, or it will dependent upon the profile only, or both profile and the alternative in question. In the SCR-dependent case, we consider only the case where the desired subset of the permutations defined specifically for each agent, alternative, profile, and the function, and introduce a characterization in terms of permutations. As we move forward in the chapter, these will become much more clearer.

7.2.1 SCR-INDEPENDENT APPROACH

First we should make our ways of analysis clearer. We seek answers to questions noted in cases below. First part is whether there exists a subset of permutations defined globally that constitutes a necessary and sufficient condition.

Definition (SCR-independent, global)**.** An $\alpha-$neutrality of $F \in \mathcal{M}$ is a permutation $\sigma \in \mathcal{P}$ with $[a \in F(R) \Rightarrow \sigma(a) \in F(\sigma(R)), \forall a \in A, \forall R \in \mathcal{R}]$. All $\alpha-$neutralities of F is denoted by $\mathcal{N}_\alpha(F)$.

The question is now, whether there exists a class of permutations $\mathcal{T} \subset \mathcal{P}$, so that:

Case 1. $F \in \mathcal{M}$, then $[F \in \mathcal{I}$ if and only if $\mathcal{T} \subset \mathcal{N}_\alpha(F)]$.

Case 2. $F \in \mathcal{M}$, then $[F \in \mathcal{I}$ if and only if $\mathcal{T} \cap \mathcal{N}_\alpha(F) \neq \emptyset]$.

The second question is whether there exists a subset of permutations defined for each profile that yields a necessary and sufficient condition.

Definition (SCR-independent, profile-wise)**.** A $\beta-$neutrality of $(F, R) \in \mathcal{M} \times \mathcal{R}$ is a permutation $\sigma \in \mathcal{P}$ with $[a \in F(R) \Rightarrow \sigma(a) \in F(\sigma(R)), \forall a \in A]$. All $\beta-$neutralities of (F, R) is denoted by $\mathcal{N}_\beta(F, R)$.

Now, does there exists a function $\mathcal{T} : \mathcal{R} \to 2^{\mathcal{P}}$ so that for given profile R, $\mathcal{T}(R)$ satisfies:

Case 3. $F \in \mathcal{M}$, then $[F \in \mathcal{I}$ if and only if $\mathcal{T}(R) \subset \mathcal{N}_\beta(F, R)$, $\forall R \in \mathcal{R}]$.

Case 4. $F \in \mathcal{M}$, then $[F \in \mathcal{I}$ if and only if $\mathcal{T}(R) \cap \mathcal{N}_\beta(F, R) \neq \emptyset$, $\forall R \in \mathcal{R}]$.

The final question is, what if the set is defined for specifically for an alternative and a profile.

Definition (SCR-independent, alternative-wise)**.** A $\theta-$neutrality of (F, a, R) $\in \mathcal{M} \times A \times \mathcal{R}$ is a permutation $\sigma \in \mathcal{P}$ with $[a \in F(R) \Rightarrow \sigma(a) \in F(\sigma(R))]$. All $\theta-$neutralities of (F, a, R) is denoted by $\mathcal{N}_\theta(F, a, R)$.

Does there exists a function $\mathcal{T} : A \times \mathcal{R} \to 2^{\mathcal{P}}$ so that for given alternative a and profile R, $\mathcal{T}(a, R)$ satisfies:

Case 5. $F \in \mathcal{M}$, then $[F \in \mathcal{I}$ if and only if $\mathcal{T}(a, R) \subset \mathcal{N}_\theta(F, a, R)$, $\forall a, R \in A \times \mathcal{R}]$.

Case 6. $F \in \mathcal{M}$, then $[F \in \mathcal{I}$ if and only if $\mathcal{T}(a, R) \cap \mathcal{N}_\theta(F, a, R) \neq \emptyset$, $\forall a, R \in A \times \mathcal{R}]$.

The answer to first five of these questions is unfortunately "no", and sixth case is left open.

Case 1. Suppose otherwise. Take any $\sigma \in \mathcal{T}$. For each $a \in A$, let F_a be the constant SCR $F_a \equiv \{a\}$. Clearly $F_a \in \mathcal{I}$. Then $\sigma \in \mathcal{T} \subset \mathcal{N}_\alpha(F_a)$. Take any $R \in \mathcal{R}$. $a \in \{a\} = F_a(R) \Rightarrow \sigma(a) \in F_a(\sigma(R)) = \{a\} \Rightarrow \sigma(a) = a$. Therefore, $\forall \sigma \in \mathcal{T}$, $a \in A$, $\sigma(a) = a$, hence $\sigma = \sigma_0$. But then $\mathcal{T} = \{\sigma_0\}$ implies that $\mathcal{M} = \mathcal{I}$, contradiction. $\square$

Case 2. Suppose otherwise. Take $f : A \to \mathbb{N}_0$ a function. Define F_f as $F_f(R) = \{x \in A : x$ is the top alternative of at least $f(x)$ people in $R\}$.

Clearly $F_f \in \mathcal{M}$. Also note that if $f(x) \leq |N| - 1$, $\forall x \in A$, then F_f satisfies NVP, which implies $F_f \in \mathcal{I}$. Particularly, consider the following case: $|A| \geq 2$, $|N| = |A|(|A|+1)/2$. Let $A = \{a_1, ..., a_{|A|}\}$. Define $f_0(a_k) = k$, $\forall k \in \{1, 2, .., |A|\}$. Since $m \geq 2$, $|N| - 1 \geq |A| \geq f_0(x)$, $\forall x \in A$. Thus $F_{f_0} \in \mathcal{I}$. Let R_0 be defined as:

$$R_0 = \begin{array}{ccccc} \overbrace{a_1}^{1 \text{ time}} & \overbrace{a_2 \ a_2}^{2 \text{ times}} & \overbrace{a_3 \ a_3 \ a_3}^{3 \text{ times}} & ... & \overbrace{a_{|A|} \ a_{|A|}..a_{|A|}}^{|A| \text{ times}} \\ .. & .. & .. & & .. \\ .. & .. & .. & & .. \end{array}.$$

Now clearly $F_{f_0}(R_0) = A$. Take any $\sigma \in \mathcal{N}_\alpha(F_{f_0})$. Since $F_{f_0}(R_0) = A$, we have $\sigma(A) = A \subset F_{f_0}(\sigma(R_0))$. Then each alternative is chosen at $\sigma(R_0)$, which implies that a_k is top alternative of at least k people in $\sigma(R_0)$. But then, we must have $\sigma(a_{|A|}) = a_{|A|} \Rightarrow \sigma(a_{|A-1|}) = a_{|A-1|} \Rightarrow ... \Rightarrow \sigma(a_1) = a_1$, i.e. $\sigma = \sigma_0$. Therefore $\mathcal{N}_\alpha(F_{f_0}) = \{\sigma_0\}$. Since $\mathcal{T} \cap \mathcal{N}_\alpha(F_{f_0}) \neq \emptyset$, we have $\sigma_0 \in \mathcal{T}$, implying that $\mathcal{M} = \mathcal{I}$, contradiction. $\square$

Case 3. Suppose otherwise. Consider a profile $R \in \mathcal{R}$. Take any $\sigma \in \mathcal{T}(R)$. For each $a \in A$, let F_a be the constant SCR $F_a \equiv \{a\}$. Clearly $F_a \in \mathcal{I}$. Then $\sigma \in \mathcal{T}(R) \subset \mathcal{N}_\alpha(F_a, R)$. Thus $a \in \{a\} = F_a(R) \Rightarrow \sigma(a) \in F_a(\sigma(R)) = \{a\}$ $\Rightarrow \sigma(a) = a$. Hence, $\sigma = \sigma_0$, implying that $\mathcal{T}(R) = \{\sigma_0\}$, $\forall R \in \mathcal{R}$. But then we get the same contradiction: $\mathcal{M} = \mathcal{I}$. $\square$

Case 4. Suppose otherwise. Let $K = \{1, 2, ..., |A|\}$. For any $x \in K^N$, "the upper partition associated with x" is the function $T^x : \mathcal{R} \to (2^A)^N$ defined as $T_i^x(R)$ is the top x_i alternatives in R_i. Any $T \in \bigcup_{x \in K^N} T^x$ is called an "upper partition". For any upper partition T and $a \in A$, define $m_a^T := |\{i \in N : a \in T_i(\bar{R})\}|$. For any $\bar{R} \in \mathcal{R}$, a feasible upper partition for $\bar{R}$ is an upper partition T such that:

1) $\forall a, b \in A$ with $a \neq b$, one has $m_a^T \neq m_b^T$,

2) At most one element of $\{m_x^T : x \in A\}$ can be larger than or equal to $|N|-1$.

We continue with the assumption that there exists a feasible upper partition for every profile. We will prove this for specific values of $|N|, |A|$ at the end of the proof. Now for each $\bar{R} \in \mathcal{R}$, fix a feasible upper partition ${}^{\bar{R}}T$, and let it be the upper partition associated with ${}^{\bar{R}}x$. Define $n_a^{\bar{R}}$ as $n_a^{\bar{R}} = m_a^{\bar{R}T}$ if $m_a^{\bar{R}T} \leq |N|-2$, and $n_a^{\bar{R}} = |N|-1$ if otherwise. Define $F_{\bar{R}}$ as $F_{\bar{R}}(R) = \{a \in A : n_a^{\bar{R}} \leq |M_a^{\bar{R}}(R)|\}$, where $M_a^{\bar{R}}(R) = \{i \in N : a \in {}^{\bar{R}}T_i(R)\}$. Now we will show that $F_{\bar{R}} \in \mathcal{M}$ and satisfies NVP, hence $F_{\bar{R}} \in \mathcal{I}$.

$F_{\bar{R}} \in \mathcal{M}$: $a \in F_{\bar{R}}(R)$ means $a \in {}^{\bar{R}}T_i(R)$ for at least $n_a^{\bar{R}}$ people, i.e. a is one the top ${}^{\bar{R}}x_i$ alternatives in R_i for at least $n_a^{\bar{R}}$ people. If R' is such that R is an $a-$refinement of R', then obviously it will still remain that way in R', i.e. $a \in F_{\bar{R}}(R')$.

$F_{\bar{R}}$ satisfies NVP: If $|N|-1$ people puts a to top place in R, since ${}^{\bar{R}}x_i \geq 1$, $\forall i \in N$, and $n_b^{\bar{R}} \leq |N|-1$, $\forall b \in A$, we get $[\exists j,\ a \in {}^{\bar{R}}T_i(R),\ \forall i \neq j \Rightarrow |M_a^{\bar{R}}(R)| \geq |N|-1 \geq n_a^{\bar{R}} \Rightarrow a \in F_{\bar{R}}(R)$. Therefore $F_{\bar{R}} \in \mathcal{I}$.

Now take any $\sigma \in \mathcal{N}_\beta(F_{\bar{R}}, \bar{R})$. Clearly, $F_{\bar{R}}(\bar{R}) = A$ since $n_a^{\bar{R}} \leq m_x^{\bar{R}T} = |M_a^{\bar{R}}(\bar{R})|$, $\forall a \in A$. Hence $A = \sigma(A) \subset F_{\bar{R}}(\sigma(\bar{R}))$. Then $\forall a \in A$, $n_a^{\bar{R}} \leq |M_a^{\bar{R}}(\sigma(\bar{R}))|$. We will prove that $\sigma = \sigma_0$.

Let $n_b^{\bar{R}} = |N|-1$ if such $b \in A$ exists. We have $n_b^{\bar{R}} \leq |M_b^{\bar{R}}(\sigma(\bar{R}))|$. Since σ is a permutation, $M_b^{\bar{R}}(\sigma(\bar{R})) = M_c^{\bar{R}}(\bar{R}) = m_c^{\bar{R}T}$ for $c = \sigma^{-1}(b)$. Then $|N|-1 \leq m_c^{\bar{R}T}$. Recall that in a feasible upper partition, at most element could have $m_x^T \geq |N|-1$, and c is indeed that element. But then, since $c \in F_{\bar{R}}(\sigma(\bar{R}))$, we have $|N|-1 = n_c^{\bar{R}} \leq |M_c^{\bar{R}}(\sigma(R))|$ implying that $b = c$, i.e. $\sigma(b) = b$.

Let $A^* = \{a \in A : n_a^{\bar{R}} = m_a^{\bar{R}T} \leq |N|-2\}$. $\forall a \in A^*$, since $a \in F_{\bar{R}}(\sigma(\bar{R}))$, we have $n_a^{\bar{R}} \leq |M_a^{\bar{R}}(\sigma(\bar{R}))| = |M_{\sigma^{-1}(a)}^{\bar{R}}(\bar{R})| = m_{\sigma^{-1}(a)}^{\bar{R}T} = n_{\sigma^{-1}(a)}^{\bar{R}}$, i.e. $n_a^{\bar{R}} \leq n_{\sigma^{-1}(a)}^{\bar{R}}$, $\forall a \in A^*$. However, we know that $\sigma(A^*) = A^*$, since $\sigma(b) = b$ even if such b exists. Therefore for every k, $\sigma^{-k}(a) \in A^*$, and there exists a k^* with

$\sigma^{-k^*}(a) = a$. Then we have, $n_a^{\bar{R}} \leq n_{\sigma^{-1}(a)}^{\bar{R}} \leq n_{\sigma^{-2}(a)}^{\bar{R}} \leq ... \leq n_{\sigma^{-k}(a)}^{\bar{R}} = n_a^{\bar{R}}$, implying that $n_a^{\bar{R}} = n_{\sigma^{-1}(a)}^{\bar{R}}$. By definiton of a feasible upper partition, we know that $n_a^{\bar{R}}$'s are pairwise different. Then $n_a^{\bar{R}} = n_{\sigma^{-1}(a)}^{\bar{R}}$ implies that $a = \sigma^{-1}(a)$, i.e. $\sigma(a) = a$. Combining both paragraphs, we obtain $\sigma(a) = a$, $\forall a \in A$.

Therefore, $\mathcal{N}_\beta(F_{\bar{R}}, \bar{R}) = \{\sigma_0\}, \forall \bar{R} \in \mathcal{R}$. Since $F_{\bar{R}} \in \mathcal{I}$, $\mathcal{T}(\bar{R}) \cap \mathcal{N}_\beta(F_{\bar{R}}, \bar{R}) \neq \emptyset$, hence $\sigma_0 \in \mathcal{T}(\bar{R})$, $\forall \bar{R} \in \mathcal{R}$. But then, again, $\mathcal{M} = \mathcal{I}$, contradiction.

What is left show the existence of a feasible upper partition for each profile. For simplicity, consider $|A| = 3$, $|N| = 4$. Without loss of generality, there are four kinds of profiles according to their first rows:

$$R^1 = \begin{bmatrix} a & a & a & a \\ & & & \\ & & & \end{bmatrix}, R^2 = \begin{bmatrix} a & a & a & b \\ & & & \\ & & & \end{bmatrix},$$

$$R^3 = \begin{bmatrix} a & a & b & b \\ & & & \\ & & & \end{bmatrix}, R^4 = \begin{bmatrix} a & a & b & c \\ & & & \\ & & & \end{bmatrix}.$$

1) Consider R^1. Either b or c passes twice in the second row. Wlog, let it be b. Take a column with a top, b second. The third place is clearly c. Then the starred region in the figure constitutes a feasible upper partition, where $x = (3, 2, 1, 1)$ and $m_a^{R^1} = 4$, $m_b^{R^1} = 2$, $m_c^{R^1} = 1$. $R^1 = \begin{bmatrix} {}^*(a) & {}^*(a) & {}^*(a) & {}^*(a) \\ {}^*(b) & {}^*(b) & & \\ {}^*(c) & & & \end{bmatrix}$.

2) Consider R^2. Take any column with a in the top. Then the starred region in the figure constitutes a feasible upper partition, where $x = (3, 1, 1, 1)$ and $m_a^{R^2} = 3$, $m_b^{R^2} = 2$, $m_c^{R^2} = 1$. $R^2 = \begin{bmatrix} {}^*(a) & {}^*(a) & {}^*(a) & {}^*(b) \\ {}^*(b \text{ or } c) & & & \\ {}^*(c \text{ or } b) & & & \end{bmatrix}$.

3) Consider R^3. Take any column with b in the top. Then the starred region in the figure constitutes a feasible upper partition, where $x = (1,1,1,3)$ and $m_a^{R^3} = 3$, $m_b^{R^3} = 2$, $m_c^{R^3} = 1$. $R^3 = \begin{bmatrix} {}^*(a) & {}^*(a) & {}^*(b) & {}^*(b) \\ & & & {}^*(a \ or \ c) \\ & & & {}^*(c \ or \ a) \end{bmatrix}$.

4) Consider R^4. Take the column with c in the top. Then the starred region in the figure constitutes a feasible upper partition, where $x = (1,1,1,3)$ and $m_a^{R^4} = 3$, $m_b^{R^4} = 2$, $m_c^{R^4} = 1$. $R^4 = \begin{bmatrix} {}^*(a) & {}^*(a) & {}^*(b) & {}^*(c) \\ & & & {}^*(a \ or \ b) \\ & & & {}^*(b \ or \ a) \end{bmatrix}$. □

Case 5. Suppose otherwise. Consider $F_a \equiv \{a\}$, the constant SCR. We have $F_a \in \mathcal{I}$, and $\mathcal{N}_\theta(F_a, a, R) = \{\sigma \in \mathcal{P} : \sigma(a) = a\} =: C_a$. Then $\mathcal{T}(a, R) \subset \bigcap_{F \in \mathcal{I}} \mathcal{N}_\theta(F, a, R) = \bigcap_{F \in \mathcal{I}} (\mathcal{N}_\theta(F, a, R) \cap C_a)$. Let R^i be a profile where i top ranks a, and all other bottom rank a. Let F^a be the following SCR:

1) $\forall b \in A$, $RC_b(F^a) = \emptyset^N$, (b is chosen at every profile)

2) $RC_a(F^a) = \{f_a(R)\} \cup \{f_a(R^i) : i \in N$ and R is not an $a-$refinement of $R^i\}$.

It is straightforward that F^a is Maskin monotonic and satisfies NVP, hence $F^a \in \mathcal{I}$. Take any $\sigma \in \mathcal{N}_\theta(F^a, a, R) \cap C_a$, i.e. $a = \sigma(a)$ and $a \in F^a(\sigma(R))$. Since $a \in F^a(\sigma(R))$, there exists an $a-$critical profile $\bar{R}$ which is an $a-$refinement of $\sigma(R)$.

If $f_a(\bar{R}) = f_a(R^i)$ for some $i \in N$, then R^i is an $a-$critical profile and also an $a-$refinement of $\sigma(R)$. Now a is at the bottom at R_i^i, a is at the top at R_j^i $\forall j \neq i$, $\sigma(a) = a$, and R^i is an $a-$refinement of $\sigma(R)$. But these together imply that R^i is an $a-$refinement of R, which is a contradiction with R^i being an $a-$critical profile.

On the other hand, if $f_a(\bar{R}) = f_a(R)$, then we have R is an $a-$critical profile and also an $a-$refinement of $\sigma(R)$. But then, since $\sigma(a) = a$, we have $\sigma(L'(a, R_i)) = L'(a, R_i)$, $\forall i \in N$. Let $D_{a,R} := \{\sigma \in \mathcal{P} : \sigma(a) = a$ and

$\sigma(L'(a, R_i)) = L'(a, R_i), \forall i \in N\}$. Then $\mathcal{T}(a, R) \subset \bigcap_{F \in \mathcal{I}} \mathcal{N}_\theta(F, a, R) \subset D_{a,R}$, however it is clear that $D_{a,R} \subset \mathcal{N}_\theta(F, a, R)$ for every Maskin monotonic F. Therefore $\mathcal{M} = \mathcal{I}$, contradiction. $\square$

7.2.2 SCR-DEPENDENT APPROACH

In this approach, we aim to find a function as before, but this time dependent on the function also. This approach can be manipulated by defining $\mathcal{T}$ as $\mathcal{T}(F) = \mathcal{N}(F)$ if $F \in \mathcal{I}$ and $\mathcal{T}(F) = \mathcal{N}(F)^c$ if $F \notin \mathcal{I}$, which would clearly do the job. However, we aim to find a meaningful $\mathcal{T}$, which is not defined in terms of the implementability of F.

We take a shortcut, and do not consider the cases which are counterparts of α, β, θ neutralities. Instead, we try to define a function also dependent on the agents. But then, a troublesome problem occurs, namely how to define a neutrality and how to apply it to a profile.

Definition. Given $a \in F(R)$, a weak neutrality of (F, a, R) is an $|N|$ tuple of permutations $\sigma = (\sigma_1, \sigma_2, .., \sigma_N) \in \mathcal{P}^N$ such that $a \in F(\sigma_1(R_1), \sigma_2(R_2), ..., \sigma_N(R_N))$. All weak neutralities of (F, a, R) is denoted by $\mathcal{N}_W(F, a, R)$.

Definition. Given $a \in F(R)$, a balanced neurality of (F, a, R) is an $|N|$ tuple of permutations $\sigma = (\sigma_1, \sigma_2, .., \sigma_N) \in \mathcal{P}^N$ such that $\exists i \in N$ with $\sigma_i(a) \in F(\sigma_1(R_1), \sigma_2(R_2), ..., \sigma_N(R_N))$. All balanced neutralities of (F, a, R) is denoted by $\mathcal{N}_B(F, a, R)$.

Weak neutrality surely is not in the spirit of neutrality since the alternative in question is kept fixed. Balanced neutrality is a more appropriate approach, however it turns out that the characterization in terms of balanced neutralities is just an extension of the characterization in terms of weak neutralities. The following definition seems the the best alternative at hand.

Definition. Given $a \in F(R)$, a neutrality of (F, a, R) is an $|N|$ tuple of permutations $\sigma = (\sigma_1, \sigma_2, .., \sigma_N) \in \mathcal{P}^N$ such that $\sigma_i(a) = \sigma_j(a)$ $\forall i, j \in N$ and

$\sigma_i(a) \in F(\sigma_1(R_1), \sigma_2(R_2), ..., \sigma_N(R_N))$. All neutralities of (F, a, R) is denoted by $\mathcal{N}(F, a, R)$.

Recall the theorem by Danilov, stating that F is Nash-implementable if and only if F is Danilov-monotonic. Danilov monotonicity is stated as:

$$a \in F(R),\ Ess(F, a, L(a, R_i)) \subset L(a, R'_i)\ \forall i \Rightarrow a \in F(R').$$

Lemma 1. *F is Danilov-monotonic (Nash-implementable) if and only if F is h^* monotonic where $h_i^F(a, R) = L(x^F(a, R_i), R_i)$ and $x^F(a, R_i)$ is the top element of $Ess(F, a, L(a, R_i))\backslash\{a\}$ in R_i.*

Proof. First part is trivial since $Ess(F, a, L(a, R_i))\backslash\{a\} \subset h_i^F(a, R)$.

For the converse, assume that F is h^* monotonic. Note that since $h_i^F(a, R) \subset L(a, R_i)$, F is Maskin monotonic. Now take any $(R, a) \in Gr(F)$. Let R' be such that $Ess(F, a, L(a, R_i)) \subset L(a, R'_i)$, $\forall i \in N$. Let R'' be derived from R so that $h_i^F(a, R)$ is reordered by putting $Ess(F, a, L(a, R_i))\backslash\{a\}$ to the bottom in some arbitrary order, and a gets down to the place just above $h_i^F(a, R)$, $\forall i \in N$. Since F is h^* monotonic, still $a \in F(R'')$. Notice that $Ess(F, a, L(a, R''_i))\backslash\{a\} \subset Ess(F, a, L(a, R_i))\backslash\{a\}$ $\forall i \in N$. Then $h_i^F(a, R'') \subset Ess(F, a, L(a, R_i))\backslash\{a\}$ since $Ess(F, a, L(a, R_i))\backslash\{a\}$ is grouped at the bottom in R''. Then consider the profile R''' derived from R'' by moving a just above $h_i^F(a, R'')$ $\forall i \in N$. By h^* monotonicity, $a \in F(R''')$. Now it is clear that $L(a, R'''_i) = h_i^F(a, R'') \cup \{a\} \subset Ess(F, a, L(a, R_i)) \subset L(a, R'_i)$ $\forall i$. Then by Maskin monotonicity, $a \in F(R')$, implying that F is Danilov-monotonic. $\square$

Definition. $\Sigma_W(F, a, R) = \prod_{i \in N} \{\sigma \in \mathcal{P} : \sigma(h_i^F(a, R)) = h_i^F(a, R)\}$.

Definition. $\Sigma_B(F, a, R) = \{\sigma \in \Sigma_W(F, a, R)$: there exists $i \in N$ with $\sigma_i(a) = a\}$.

Theorem 3. *F is Nash-implementable if and only if F is Maskin monotonic and $\Sigma_W(F, a, R) \subset \mathcal{N}_W(F, a, R)$, $\forall (R, a) \in Gr(F)$.*

Proof. Assume that F is Nash-implementable, i.e. Danilov-monotonic. Take any $(R, a) \in Gr(F)$ and $\sigma \in \Sigma_W(F, a, R)$, let $R'_i = \sigma_i(R_i)$ $\forall i \in N$. Note that $a \notin h_i^F(a, R)$. From definition we have $aR'_i h_i^F(a, R) \Rightarrow aR'_i Ess(F, a, L(a, R_i))$ $\forall i \in N$. Then $a \in F(R')$ by Danilov-monotonicity. Therefore $\sigma \in \mathcal{N}_W(F, a, R)$, hence $\Sigma_W(F, a, R) \subset \mathcal{N}_W(F, a, R)$, $\forall (R, a) \in Gr(F)$.

For the converse, assume that F is Maskin monotonic and $\Sigma_W(F, a, R) \subset \mathcal{N}_W(F, a, R)$, $\forall (R, a) \in Gr(F)$. Take any $(R, a) \in Gr(F)$. Assume that R' satisfies $h_i^F(a, R) \subset L(a, R'_i)$ $\forall i$. Take a permutation $\sigma' \in \mathcal{P}^N$ such that $\sigma'_i(h_i^F(a, R)) = h_i^F(a, R)$ and $\sigma'_i(A \backslash h_i^F(a, R))$ is ordered as in R'_i, $\forall i$. It is clear that $\sigma' \in \Sigma_W(F, a, R) \subset \mathcal{N}_W(F, a, R)$, hence $a \in F(R'')$, where $R'' = (\sigma'_1(R_1), \sigma'_2(R_2), ..., \sigma'_N(R_N))$. But note that $L(a, R'_i) = L(a, R''_i)$ $\forall i$. Then by Maskin monotonicity of F, $a \in F(R')$, hence F is h^* monotonic, hence Nash-implementable by the lemma. □

Theorem 4. *F is Nash-implementable if and only if F is Maskin monotonic and $\Sigma_B(F, a, R) \subset \mathcal{N}_B(F, a, R)$, $\forall (R, a) \in Gr(F)$.*

Proof. Assume that F is Nash-implementable, i.e. Danilov-monotonic. Take any $(R, a) \in Gr(F)$ and $\sigma \in \Sigma_B(F, a, R)$, let $R'_i = \sigma_i(R_i)$ $\forall i \in N$. $\sigma \in \Sigma_B(F, a, R) \subset \Sigma_W(F, a, R) \subset \mathcal{N}_W(F, a, R)$, hence $a \in F(R')$. Since $\sigma \in \Sigma_B(F, a, R)$, $\exists j \in N$ with $\sigma_j(a) = a$, i.e. $\exists j \in N$ with $\sigma_j(a) \in F(R')$. Therefore $\sigma \in \mathcal{N}_B(F, a, R)$, hence $\Sigma_B(F, a, R) \subset \mathcal{N}_B(F, a, R)$, $\forall (R, a) \in Gr(F)$.

For the converse, assume that F is Maskin monotonic and $\Sigma_B(F, a, R) \subset \mathcal{N}_B(F, a, R)$, $\forall (R, a) \in Gr(F)$. Assume that R' satisfies $h_i^F(a, R) \subset L(a, R'_i)$ $\forall i$. Let the alternative just above $x^F(a, R_i)$ in R_i be called $y^F(a, R_i)$. Take $\sigma' \in \mathcal{P}^N$ such that $[\sigma'_1 = \sigma_0]$ and $\forall i \neq 1$, $[\sigma'_i(h_i^F(a, R)) = h_i^F(a, R)$, $\sigma'_i(a) = y^F(a, R_i)$, $\sigma'_i(y^F(a, R_i)) = a$, and other elements are kept fixed$]$. Now it is clear that $\sigma' \in \Sigma_B(F, a, R) \subset \mathcal{N}_B(F, a, R)$, hence $\exists j \in N$ with $\sigma'_j(a) \in F(R'')$, where $R'' = (\sigma_1(R_1), \sigma_2(R_2), ..., \sigma_N(R_N))$. Notice that $\{\sigma'_i(a) : i \in N\} = \{a\} \cup \{y^F(a, R_i) : i \neq 1\}$. If it was the case that $y^F(a, R_i) \in F(R'')$ and $a \neq y^F(a, R_i)$ for some $i \neq 1$, then $y^F(a, R_i)$ would be an essential element other than a. However, $y^F(a, R_i) \notin h_i^F(a, R) \supset Ess(F, a, L(a, R_i))\backslash\{a\}$. Contradiction. Therefore $a \in F(R'')$. Now, similarly take $\sigma'' \in \mathcal{P}^N$ such that $[\sigma_i'' = \sigma_0 \ \forall i \neq 1]$ and $[\sigma_1''(h_1^F(a, R'')) = h_1^F(a, R'')$, $\sigma_1''(a) = y^F(a, R_1'')$, $\sigma_1''(y^F(a, R_1'')) = a$, and other elements are kept fixed]. Then we have $\sigma'' \in \Sigma_B(F, a, R'') \subset \mathcal{N}_B(F, a, R'')$, hence $\exists j \in N$ with $\sigma_j'(a) \in F(R''')$, where $R''' = (\sigma_1(R_1''), \sigma_2(R_2''), ..., \sigma_N(R_N''))$. With the same argument we obtain $a \in F(R''')$. But then by Maskin monotonicity, $a \in F(R')$. Therefore, F is h^* monotonic, i.e. Nash-implementable. $\square$

CHAPTER 8

CONCLUSION

Refined center makes it a lot easier to write down a specific SCR. Just by writing down the critical profiles for each alternative, we may formally talk of the SCR. Moreover, it should be noted that U^T is independent of SCR in question. It makes well possible to track down to all SCR's that have T as a center by following the steps. The result may lead to further research in the sense that there may occur other common properties of the SCR's that have a common center. Moreover, by the fact that any subset of a center is also a center, we may talk of largest centers. It would be interesting to find largest centers, and investigate what other common properties the SCR's with the largest centers have. Finally in chapter 1, we have proved that the whole set of preference profiles can never be a center. Therefore it is -always- possible to find an SCR by using a proper subset of the set all profiles.

Concerning the self-monotonicity, we have proved that the set of self-monotonicities of an SCR is a cartesian product of some sets associated with each element of the graph of the SCR. The fact that it is a cartesian product brings about the idea that there is a form of neutrality of subsets of the alternatives in self-monotonicities. The main result of chapter 2 is that there exists a lattice structure behind the monotonicities, and various self-monotonicities could be derived from a monotonicity. The image of the function f has some -chosen- number of elements, and that allows to partition the self-monotonicities with respect to the number of (i, b) couples that the self-monotonicity associates to a single profile in $R^(*)$.

In chapter 3, if one would ask what are the most natural candidates that preserve Maskin-Monotonicity, the most natural and also trivial answer would be union, intersection, left operator, right operator, empty set, and a constant set. It is interesting that the whole class of operations that preserve Maskin-Monotonicity consists of exactly these operations but only differently applied for each alternative. Moreover, these operations are closely related to semi-group theory which brings about another research topic. Furthermore, if not all but some of these operations are allowed, such as only union intersection and left operator, these operations are transitive and there occurs a partial order on the class of Maskin-Monotonic SCR's. Hence we may talk of maximal elements and equivalence classes.

The results concerning the impossibility and possibility domains are quite interesting. It turns out that when we employ Manhattan metric, i.e. by adding 1 for each transpositions, we miss out the information that the bound n is indeed $(n-1) * 1 + 1$ and that last 1 is quite important than $n-1$ 1's. That last 1 corresponds to the transposition of second and third place. The main of reason of the impossibility is switching second and third place even after taking the least element in the social norm to top place. In order to get rid of the impossibility, the domain should satisfy the consolation property, namely the social center should be appeased by any individual that top ranks the bottom of the norm, by putting the top alternative of the social norm to the second place. Moreover, depending on the weights, it may become well possible that there are more than one impossibility and possibility domains, which are consecutively nested, particularly, there exists impossibility domains which are subsets of some possibility domains. Therefore, depending on the state of society and the domain, an limited increment in the freedom of the society may rule out the impossibility result, resulting in the feasibility of a desired social choice rule.

Although we could not reach the results we aimed for in the last chapter, the results are still telling about neutrality. First of all, SCR's that are Maskin-Monotonic and neutral with respect to transpositions, say transpositive, are still Nash-implementable. It seems that Maskin-Monotonicity should also be modified in order to reach an original necessary and sufficient condition in terms of neutrality. Otherwise, the best way at hand would be just recasting Danilov's work. Also, the examples with the upper partitions tell us that some SCR's which are Nash-implementable are not even neutral between two alternatives. They behave each and every alternative differently but they are still Nash-implementable. Of course that is the case when we do not consider different permutations for different voters. Therefore, it seems that Nash-implementability could be more related to rights of voters than the rights of alternatives, consistent with the NVP approach.

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